

# A DIRECT APPROACH TO THE BISPECTRAL PROBLEM FOR THE RUIJSENAARS-MACDONALD $q$ -DIFFERENCE OPERATORS

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ABSTRACT. We present a direct approach to the bispectral problem associated with the Ruijsenaars-Macdonald  $q$ -difference operators of  $GL$  type. We give an explicit construction of the meromorphic function  $\psi_n(x; s|q, t)$  on  $\mathbb{T}_x^n \times \mathbb{T}_s^n$ , which is the solution of the bispectral problem up to a certain gauge transformation. Some basic properties of  $\psi_n(x; s|q, t)$  are studied, including the structure of the divisor of poles, the symmetries  $\psi_n(x; s|q, t) = \psi_n(s; x|q, t) = \psi_n(x; s|q, q/t)$ , and the recursive structure described in terms of Jackson integrals or  $q$ -difference operators.

## 1. INTRODUCTION

In this paper we study the bispectral problem for the Ruijsenaars-Macdonald  $q$ -difference operators of  $GL$  type. Let  $\mathbb{T}_x^n = (\mathbb{C}^*)^n$  be the  $n$ -dimensional algebraic torus with canonical coordinates  $x = (x_1, \dots, x_n)$ . For each  $r = 0, 1, \dots, n$ , we denote by  $D_r^x = D_r(x, T_{q,x}|t)$  the Ruijsenaars-Macdonald  $q$ -difference operator of order  $r$  in  $x$  variables with parameter  $t \in \mathbb{C}$ ; it is defined by

$$D_r^x = t^{\binom{n}{2}} \sum_{|I|=r} \prod_{i \in I; j \notin I} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i} \quad (r = 0, 1, \dots, n), \quad (1.1)$$

summed over all subsets  $I \subseteq \{1, \dots, n\}$  of cardinality  $r$ , where  $T_{q,x_i}$  stands for the  $q$ -shift operator with respect to  $x_i$  ( $i = 1, \dots, n$ ). Denoting the dual coordinates by  $s = (s_1, \dots, s_n)$ , we investigate the joint bispectral problem

$$\begin{aligned} D_r^x f(x; s) &= f(x; s) e_r(s) & (r = 1, \dots, n), \\ D_r^s f(x; s) &= f(x; s) e_r(x) & (r = 1, \dots, n), \end{aligned} \quad (1.2)$$

for an unknown meromorphic function  $f(x; s)$  on  $\mathbb{T}_x^n \times \mathbb{T}_s^n$ . Here, for each  $r = 0, 1, \dots, n$ ,  $e_r(s)$  denotes the elementary symmetric polynomial in  $s$  variables of degree  $r$ . We also write this system (1.2) of  $q$ -difference equations as

$$\begin{aligned} D^x(u) f(x; s) &= f(x; s) \prod_{i=1}^n (1 - us_i), \\ D^s(u) f(x; s) &= f(x; s) \prod_{i=1}^n (1 - ux_i), \end{aligned} \quad (1.3)$$

by using the generating function  $D^x(u) = \sum_{r=0}^n (-u)^r D_r^x$  of the Ruijsenaars-Macdonald operators.

It is known by Ruijsenaars [9] and Macdonald [4] that the  $q$ -difference operators  $D_1^x, \dots, D_n^x$  commute with each other. On the ring  $\mathbb{C}[x]^{\mathfrak{S}_n}$  of symmetric polynomials

in  $x$ , they are diagonalized simultaneously by the Macdonald polynomials  $P_\lambda(x|q, t)$  associated with partitions  $\lambda = (\lambda_1, \dots, \lambda_n)$ :

$$D^x(u) P_\lambda(x|q, t) = P_\lambda(x|q, t) \prod_{i=1}^n (1 - ut^{n-i} q^{\lambda_i}). \quad (1.4)$$

It is also known that the Macdonald polynomials  $\tilde{P}_\lambda(x|q, t) = P_\lambda(x|q, t)/P_\lambda(t^\delta|q, t)$  normalized by the values at  $t^\delta = (t^{n-1}, t^{n-2}, \dots, 1)$  have the duality property

$$\tilde{P}_\lambda(t^\delta q^\mu|q, t) = \tilde{P}_\mu(t^\delta q^\lambda|q, t) \quad (1.5)$$

for any partitions  $\lambda, \mu$  of length  $\leq n$ . This duality implies that the function  $f(x; s) = \tilde{P}_\lambda(t^\delta q^\mu|q, t)$  solves the bispectral problem (1.3) on the discrete set of points  $(x; s) = (t^\delta q^\mu, t^\delta q^\lambda)$  indexed by partitions.

The bispectral problem (1.3) for the Ruijsenaars-Macdonald operators in complex variables is studied by Cherednik in the context of difference Harish-Chandra theory [1, 2], and by van Meer and Stokman [6] in connection with the quantum KZ equations. The existence of meromorphic solutions has also been established by their works. In this article, we present a direct approach to the bispectral problem in the region  $|x_1| \gg |x_2| \gg \dots \gg |x_n|$  and  $|s_1| \gg |s_2| \gg \dots \gg |s_n|$ , based on formal solutions of the  $q$ -difference equations (1.3) of scalar type. In particular we give explicit construction of meromorphic solutions, and discuss various properties of solutions, including duality and recursive constructions by Jackson integral representations and  $q$ -difference operators.

First, we establish several fundamental properties concerning the joint eigenfunction problem with respect to the Ruijsenaars-Macdonald operators acting on  $x$

$$D^x(u) f(x; s) = f(x; s) \prod_{i=1}^n (1 - us_i), \quad (1.6)$$

with  $s = (s_1, \dots, s_n)$  being a given set of complex variables. Set  $s_i = t^{n-i} q^{\lambda_i}$  ( $1 \leq i \leq n$ ). Let  $Q_+ = \mathbb{N}\alpha_1 \oplus \dots \oplus \mathbb{N}\alpha_{n-1}$  be the positive cone of the root lattice of  $A_{n-1}$  with simple roots  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  ( $1 \leq i \leq n-1$ ). We write  $x^\lambda = \prod_i x_i^{\lambda_i}$ ,  $x^{-\mu} = x_1^{-\mu_1} \dots x_n^{-\mu_n}$  for each  $\mu = \sum_i \mu_i \epsilon_i \in Q_+$ , and  $\mathbb{C}(s^{-Q_+})[[x^{-Q_+}]] = \mathbb{C}(s_2/s_1, \dots, s_n/s_{n-1})[[x_2/x_1, \dots, x_n/x_{n-1}]]$  for short. Let  $f(x; s)$  be a formal series solutions for the joint eigenfunction problem (1.6) given by  $f(x; s) = x^\lambda \varphi(x; s)$ ,  $\varphi(x; s) = \sum_{\mu \in Q_+} x^{-\mu} \varphi_\mu(s) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$  having the initial term  $\varphi_0(s) = 1$ .

**Theorem 1.1** (Theorem 2.2). (1) *Such a series  $\varphi(x; s)$  exists uniquely.* (2) *For each  $\mu \in Q_+$  the coefficient  $\varphi_\mu(s) \in \mathbb{C}(s^{-Q_+})$  is regular at the origin  $(s_2/s_1, \dots, s_n/s_{n-1}) = 0$ .* (3) *Each  $\varphi_\mu(s)$  has at most simple poles along the divisors  $q^{k+1}s_j/s_i = 1$  ( $1 \leq i < j \leq n; k = 0, 1, 2, \dots$ ).*

Set  $\psi(x; s)$  via the gauge transformation

$$\varphi(x; s) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \psi(x; s), \quad (1.7)$$

where we used the standard notation for the  $q$ -shifted factorial  $(z; q)_\infty = \prod_{k=0}^\infty (1 - q^k z)$  assuming that  $|q| < 1$ . Then (1.6) is recast as

$$L^{(x;s)}(u)\psi(x; s) = \psi(x; s) \prod_{i=1}^n (1 - us_i), \quad (1.8)$$

where

$$\begin{aligned} L^{(x;s)}(u) &= \sum_{I \subseteq \{1, \dots, n\}} (-u)^{|I|} s^{\epsilon_I} B_I(x) T_{q,x}^{\epsilon_I} \quad (\epsilon_I = \sum_{i \in I} \epsilon_i), \\ B_I(x) &= \prod_{i < j; i \notin I, j \in I} \frac{1 - tx_j/x_i}{1 - x_j/x_i} \frac{1 - qx_j/tx_i}{1 - qx_j/x_i} \quad (I \subseteq \{1, \dots, n\}). \end{aligned} \quad (1.9)$$

Since  $L^{(x;s)}(u)$  is invariant under the exchange  $t \leftrightarrow q/t$ , we have

**Theorem 1.2** (Theorem 2.4). *The series  $\psi(x; s)$  is invariant under the change of parameters  $t \leftrightarrow q/t$ .*

Now we turn to an explicit formula for the eigenfunction of (1.6). For each  $n = 1, 2, \dots$ , we denote by  $M_n$  the set of all strictly upper triangular  $n \times n$  matrices with nonnegative integer coefficients:  $M_n = \{\theta = (\theta_{ij})_{i,j=1}^n \in \text{Mat}(n; \mathbb{N}) \mid \theta_{ij} = 0 \ (1 \leq j \leq i \leq n)\}$ . For each  $\theta \in M_n$ , we define a rational function  $c_n(\theta; s|q, t)$  in the variables  $s = (s_1, \dots, s_n)$  by

$$\begin{aligned} c_n(\theta; s|q, t) &= \prod_{k=2}^n \prod_{1 \leq i < j \leq k} \frac{(q^{\sum_{a>k} (\theta_{i,a} - \theta_{j,a})} t s_j / s_i; q)_{\theta_{i,k}}}{(q^{\sum_{a>k} (\theta_{i,a} - \theta_{j,a})} q s_j / s_i; q)_{\theta_{i,k}}} \\ &\quad \cdot \prod_{k=2}^n \prod_{1 \leq i \leq j < k} \frac{(q^{-\theta_{j,k} + \sum_{a>k} (\theta_{i,a} - \theta_{j,a})} q s_j / t s_i; q)_{\theta_{i,k}}}{(q^{-\theta_{j,k} + \sum_{a>k} (\theta_{i,a} - \theta_{j,a})} s_j / s_i; q)_{\theta_{i,k}}}, \end{aligned} \quad (1.10)$$

where  $(z; q)_k = (z; q)_\infty / (q^k z; q)_\infty$  ( $k \in \mathbb{Z}$ ), and  $\sum_{a>k} = \sum_{a=k+1}^n$ . Set

$$p_n(x; s|q, t) = \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j / x_i)^{\theta_{ij}} \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]. \quad (1.11)$$

**Theorem 1.3** (Theorem 3.3). *We have the eigenfunction equation in  $x$*

$$D^x(u) x^\lambda p_n(x; s|q, t) = x^\lambda p_n(x; s|q, t) \prod_{i=1}^n (1 - us_i). \quad (1.12)$$

In Section 3, this will be proved by using the property of  $c_n(\theta; s|q, t)$  and the theory of Macdonald polynomials. We note that an alternative proof which does not rely on the Macdonald theory will be presented in Sections 4 and 5 based on a recursive use of certain  $q$ -difference operators.

Now we embark on the study of the bispectral problem (1.2). Let  $e_n(x; s)$  be a (possibly multi-valued) meromorphic function in  $x, s$  such that

$$T_{q,x_i} e_n(x; s) = e_n(x; s) s_i t^{-n+i}, \quad T_{q,s_i} e_n(x; s) = e_n(x; s) x_i t^{-n+i} \quad (1 = 1, \dots, n), \quad (1.13)$$

and  $e_n(x; s) = e_n(s, x)$ . Set

$$\varphi_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} p_n(x; s|q, t), \quad (1.14)$$

$$\begin{aligned} \psi_n(x; s|q, t) &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} p_n(x; s|q, t) \\ &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty} \varphi_n(x; s|q, t), \end{aligned} \quad (1.15)$$

$$f_n(x; s|q, t) = e_n(x; s) \varphi_n(x; s|q, t) \in e_n(x; s) \mathbb{C}[[s^{-Q_+}]] [[x^{-Q_+}]]. \quad (1.16)$$

By a recursive use of certain Jackson integral transformations studied in [7], we have

**Theorem 1.4** (Theorem 4.4). *The series  $f_n(x; s|q, t)$  is a formal solution of the bispectral problem (1.2) associated with the Ruijsenaars-Macdonald operators. We have the symmetries*

$$\varphi_n(x; s|q, t) = \varphi_n(s; x|q, t), \quad (1.17)$$

$$\varphi_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \frac{(ts_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} \varphi_n(x; s|q, q/t), \quad (1.18)$$

$$\psi_n(x; s|q, t) = \psi_n(s; x|q, t), \quad (1.19)$$

$$\psi_n(x; s|q, t) = \psi_n(x; s|q, q/t). \quad (1.20)$$

We can recast the recurrence of the Jackson integrals by that of a certain  $q$ -difference operators. Note that such a class of  $q$ -difference operators, called the Ruijsenaars-Macdonald operators of *row type*, is studied in [8]. Set

$$K^{(x; s|q, t)}(u) = \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n (u/s_i)^{\nu_i} \prod_{i=1}^n \frac{(t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \frac{(q^{-\nu_j+1}x_j/tx_i; q)_{\nu_i}}{(q^{-\nu_j}x_j/x_i; q)_{\nu_i}} T_{q, x}^{-\nu}. \quad (1.21)$$

**Theorem 1.5** (Theorem 5.1). *We have the recurrence relations*

$$\begin{aligned} \psi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(tx_{n+1}/x_i; q)_\infty}{(qx_{n+1}/x_i; q)_\infty} \frac{(qs_{n+1}/ts_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \\ &\quad \cdot K^{(x; s|q, t)}(qs_{n+1}/t) K^{(s; x|q, q/t)}(tx_{n+1}) \psi_n(s; x|q, t) \quad (n = 1, 2, \dots). \end{aligned} \quad (1.22)$$

This provides us with an alternative proof of Theorem 1.4 which does not depend on the theory of Macdonald polynomials.

Finally, let  $e_n(x; s) \varphi_n(x; s|q, t)$  be our formal solution of the bispectral problem for the Ruijsenaars-Macdonald operators, and set

$$F_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} (qx_j/tx_i)_\infty (qs_j/ts_i)_\infty \varphi_n(x; s|q, t). \quad (1.23)$$

**Theorem 1.6** (Theorem 6.5). *The series  $F_n(x; s|q, t)$  represents a holomorphic function on  $\mathbb{C}_z^{n-1} \times \mathbb{C}_w^{n-1}$  in the variables  $(z; w) = (z_1, \dots, z_{n-1}; w_1, \dots, w_{n-1})$  with  $z_i = x_{i+1}/x_i, w_i = s_{i+1}/s_i$  ( $i = 1, \dots, n-1$ ), depending holomorphically on  $t \in \mathbb{C}^*$  such that  $F_n(x; s|q, t) = F_n(s; x|q, t) = F_n(x; s|q, q/t)$ .*

As an application of our results, we have a summation formula which can be regarded as an infinite series version of the principal substitution of the Macdonald polynomial  $P_\lambda(t^\delta|q, t)$ .

**Theorem 1.7** (Theorem 6.6). *Let  $|t| > |q|^{-(n-2)}$ . We have*

$$\sum_{\theta \in M_n} c_n(\theta; s|q, t) t^{\sum_{i < j} (i-j)\theta_{ij}} = \prod_{i=1}^n \frac{(q/t; q)_\infty}{(q/t^i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty}. \quad (1.24)$$

In view of the explicit construction of  $\varphi_n(x; s|q, t)$  as above, one observes that the properties stated in Theorems 1.1 and 1.4 are hidden and not evident at all. It seems a challenging problem to have an alternative expression of  $\varphi_n(x; s|q, t)$  in which such properties are manifest. At least the cases  $n = 2$  and  $3$  can be worked out. The case  $n = 2$  is easy and given in (4.11).

**Theorem 1.8** (Theorem 7.2). *We have*

$$\begin{aligned} \varphi_3(x, s|q, t) &= \sum_{k \geq 0} \frac{(q/t; q)_k (q/t; q)_k}{(q; q)_k (t; q)_k} (qx_3s_3/x_1s_1)^k \\ &\times \prod_{1 \leq i < j \leq 3} \frac{(t; q)_\infty (qx_js_j/x_is_i; q)_\infty}{(qx_j/tx_i; q)_\infty (qs_j/ts_i; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} qx_j/tx_i, qs_j/ts_i \\ qx_js_j/x_is_i \end{matrix}; q, q^kt \right] \quad (|t| < 1), \end{aligned} \quad (1.25)$$

which manifestly shows the regularity in Theorem 1.1, and the duality  $\varphi_3(x, s|q, t) = \varphi_3(s, x|q, t)$ .

As yet another application, we will revisit the problem considered in [11], and examine some properties of a certain family of integral transformations called  $I(\alpha)$  ( $\alpha \in \mathbb{C}$ ) (see Section 8). We claim that we have the commutativity  $[D^x(u), I(\alpha)] = 0$  and  $[I(\alpha), I(\beta)] = 0$ .

Organization of the paper is as follows. In Section 2, we establish several fundamental properties of formal solutions for the joint eigenfunction problem associated with the Ruijsenaars-Macdonald operators in the variables  $x$ . In Section 3, we study the properties of the explicit formulas for the eigenfunction. In Section 4, the recursive structure in the explicit formula is investigated from the point of view of the Jackson integrals, which enables us to study the eigenfunction problem associated with the Ruijsenaars-Macdonald operators in the variables  $s$ . In Section 5, the recursive structure is recast in yet another form based on a recursive application of certain  $q$ -difference operators, thereby establishing the bispectral eigenfunction problem associated with the Ruijsenaars-Macdonald operators in the pair of variables  $x$  and  $s$ , without relying on the theory of Macdonald polynomials. Section 6 is devoted to the study of the convergence of the formal series solutions. In Section 7, we treat the case  $n = 3$  and obtain some formulas for the eigenfunctions in which the duality can be seen manifestly. As an application of our result, we study the family of integral transformations  $I(\alpha)$  ( $\alpha \in \mathbb{C}$ ) in Section 8.

Throughout this paper, we fix nonzero constants  $q, t \in \mathbb{C}^*$  with the assumption  $|q| < 1$ . We use the standard notation of  $q$ -shifted factorials

$$(z; q)_\infty = \prod_{k=0}^{\infty} (1 - q^k z), \quad (z; q)_k = \frac{(z; q)_\infty}{(q^k z; q)_\infty} \quad (k \in \mathbb{Z}), \quad (1.26)$$

theta function  $\theta(z; q) = (z; q)_\infty (q/z; q)_\infty (q; q)_\infty$ , and Jackson integrals

$$\begin{aligned} \int_0^a f(z) \frac{d_q z}{z} &= (1 - q) \sum_{k=0}^{\infty} f(q^k a), \quad \int_a^\infty f(z) \frac{d_q z}{z} = (1 - q) \sum_{k=0}^{\infty} f(q^{-k-1} a), \\ \int_0^{a^\infty} f(z) \frac{d_q z}{z} &= \int_0^a f(z) \frac{d_q z}{z} + \int_a^\infty f(z) \frac{d_q z}{z} = (1 - q) \sum_{k=-\infty}^{\infty} f(q^k a). \end{aligned} \quad (1.27)$$

The basic hypergeometric series  ${}_{r+1}\phi_r(a_1, a_2, \dots, a_{r+1}; b_1, \dots, b_r; q, z)$  ( $r = 0, 1, \dots$ ) is defined by

$${}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, q)_n (a_2, q)_n \cdots (a_{r+1}, q)_n}{(q, q)_n (b_1, q)_n \cdots (b_r, q)_n} z^n. \quad (1.28)$$

We also use the notation for the very well-poised series as

$$\begin{aligned} {}_{r+1}W_r(a_1; a_4, a_5, \dots, a_{r+1}; q, z) \\ = {}_{r+1}\phi_r \left[ \begin{matrix} a_1, qa_1^{1/2} - qa_1^{1/2}, a_4, \dots, a_{r+1} \\ a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix}; q, z \right]. \end{aligned} \quad (1.29)$$

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## 2. FORMAL SOLUTIONS OF THE EIGENFUNCTION EQUATION

In this section we investigate *formal solutions*  $f(x; s)$  of the eigenfunction equation

$$D^x(u) f(x; s) = f(x; s) \prod_{i=1}^n (1 - us_i) \quad (2.1)$$

in the variables  $x = (x_1, \dots, x_n)$ , and prove the unique existence of a formal solution  $f(x; s)$  with a given leading coefficient. We also establish certain regularity of coefficients of  $f(x; s)$  with respect to  $s = (s_1, \dots, s_n)$ , and symmetry of solutions under the change of parameters  $t \leftrightarrow q/t$ .

We first introduce some notations in order to clarify the meaning of “formal solutions”. Let  $P = \mathbb{Z}\epsilon_1 \oplus \dots \oplus \mathbb{Z}\epsilon_n$  be the free  $\mathbb{Z}$ -module with canonical basis  $\{\epsilon_1, \dots, \epsilon_n\}$ , and  $\langle \cdot, \cdot \rangle : P \times P \rightarrow \mathbb{Z}$  the symmetric bilinear form defined by  $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$  ( $i, j \in \{1, \dots, n\}$ ). We denote by

$$Q_+ = \mathbb{N}\alpha_1 \oplus \dots \oplus \mathbb{N}\alpha_{n-1} \subseteq P \quad (\mathbb{N} = \{0, 1, 2, \dots\}) \quad (2.2)$$

the cone generated by the simple roots  $\alpha_i = \epsilon_i - \epsilon_{i+1}$  ( $i = 1, \dots, n-1$ ). Under the identification  $P \xrightarrow{\sim} \mathbb{Z}^n$ , we use the multi-index notation of monomials  $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}$  and  $q$ -difference operators  $T_{q,x}^\mu = T_{q,x_1}^{\mu_1} \dots T_{q,x_n}^{\mu_n}$  for each  $\mu = \sum_{i=1}^n \mu_i \epsilon_i = (\mu_1, \dots, \mu_n) \in P$ , so that  $T_{q,x}^\mu(x^\nu) = q^{\langle \mu, \nu \rangle} x^\nu$  for any  $\mu, \nu \in P$ . We use the notations

$$\mathbb{C}[[x^{-Q_+}]] = \mathbb{C}[[x_2/x_1, \dots, x_n/x_{n-1}]] \quad \text{and} \quad \mathbb{C}(x^{-Q_+}) = \mathbb{C}(x_2/x_1, \dots, x_n/x_{n-1}) \quad (2.3)$$

for the ring of formal power series and the field of rational functions in  $(n-1)$  variables  $x^{-\alpha_i} = x_{i+1}/x_i$  ( $i = 1, \dots, n-1$ ), respectively. When we work with these algebras, through the identification  $z_i = x_{i+1}/x_i$  ( $i = 1, \dots, n-1$ ), we use  $(x_2/x_1, \dots, x_n/x_{n-1})$  as a conventional notation for the canonical coordinates  $z = (z_1, \dots, z_{n-1})$  of an  $(n-1)$ -dimensional affine space  $\mathbb{C}_z^{n-1}$ . By this convention, we interpret the expression  $x_j/x_i$  ( $i < j$ ) as representing the monomial  $z_i z_{i+1} \dots z_{j-1}$ .

**2.1. Formal solution with a given leading coefficient.** We consider formal solutions  $f(x; s)$  of (2.1) in the form

$$f(x; s) = x^\lambda \varphi(x; s), \quad \varphi(x; s) = \sum_{\mu \in Q_+} x^{-\mu} \varphi_\mu(s), \quad (2.4)$$

where  $\lambda = (\lambda_1, \dots, \lambda_n)$  denotes the complex variables such that  $s_i = t^{n-i} q^{\lambda_i}$  ( $i = 1, \dots, n$ ). As far as  $q$ -difference equations are concerned, the power function  $x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}$  above may be replaced by any function  $e(x; s)$  such that  $T_{q,x_i} e(x; s) = e(x; s) q^{\lambda_i} = e(x; s) s_i / t^{n-i}$  ( $i = 1, \dots, n$ ). In the expansion (2.4) of  $\varphi(x; s)$ , we assume that all the coefficients  $\varphi_\mu(s)$  belong either to the ring  $\mathbb{C}[[s^{-Q_+}]]$  of formal power series, or to the ring  $\mathbb{C}(s^{-Q_+})$  of rational functions in the variables  $(s_2/s_1, \dots, s_n/s_{n-1})$ . The coefficient  $\varphi_0(s)$  is called the *leading coefficient* of the solution  $f(x; s)$ , or the *initial value* of  $\varphi(x; s)$  at the origin  $(x_2/x_1, \dots, x_n/x_{n-1}) = 0$ , depending on the situation.

**Theorem 2.1** (Unique existence of a formal solution). *Let  $\lambda = (\lambda_1, \dots, \lambda_n)$  be the complex variables such that  $s_i = t^{n-i} q^{\lambda_i}$  for  $i = 1, \dots, n$ . Then, for any  $c(s) \in \mathbb{C}[[s^{-Q_+}]]$*



(resp.  $\in \mathbb{C}(s^{-Q_+})$ ), there exists a unique formal solution

$$f(x; s) = x^\lambda \varphi(x; s) \in x^\lambda \mathbb{C}[[s^{-Q_+}]] [[x^{-Q_+}]] \quad (\text{resp. } \in x^\lambda \mathbb{C}(s^{-Q_+}) [[x^{-Q_+}]]) \quad (2.5)$$

of (2.1) with leading coefficient  $\varphi_0(s) = c(s)$ .

**Theorem 2.2** (Formal solution with rational coefficients). *With the variables  $\lambda = (\lambda_1, \dots, \lambda_n)$  such that  $s_i = t^{n-i} q^{\lambda_i}$  ( $i = 1, \dots, n$ ), let*

$$f(x; s) = x^\lambda \varphi(x; s) \in x^\lambda \mathbb{C}(s^{-Q_+}) [[x^{-Q_+}]], \quad \varphi(x; s) = \sum_{\mu \in Q_+} x^{-\mu} \varphi_\mu(s), \quad (2.6)$$

be the formal solution of (2.1) with leading coefficient  $\varphi_0(s) = 1$ . Then, for each  $\mu \in Q_+$ , the coefficient  $\varphi_\mu(s) \in \mathbb{C}(s^{-Q_+})$  is regular at the origin  $(s_2/s_1, \dots, s_n/s_{n-1}) = 0$ , and has at most simple poles along the divisors

$$q^{k+1} s_j / s_i = 1 \quad (1 \leq i < j \leq n; k = 0, 1, 2, \dots). \quad (2.7)$$

In order to prove these theorems, we rewrite the eigenfunction equation for  $f(x; s)$  to that of  $\varphi(x; s)$ . The  $q$ -difference operator

$$E^{(x;s)}(u) = x^{-\lambda} D^x(u) x^\lambda \quad (2.8)$$

defined by conjugation is expressed explicitly as

$$\begin{aligned} E^{(x;s)}(u) &= \sum_{I \subseteq \{1, \dots, n\}} (-u)^{|I|} s^{\epsilon_I} A_I(x) T_{q,x}^{\epsilon_I}, \\ A_I(x) &= \prod_{i < j; i \in I, j \notin I} \frac{1 - x_j / tx_i}{1 - x_j / x_i} \prod_{i < j; i \notin I, j \in I} \frac{1 - tx_j / x_i}{1 - x_j / x_i} \quad (I \subseteq \{1, \dots, n\}), \end{aligned} \quad (2.9)$$

where  $\epsilon_I = \sum_{i \in I} \epsilon_i$ . With this  $q$ -difference operator, the eigenfunction equation for  $\varphi(x; s)$  is given by

$$E^{(x;s)}(u) \varphi(x; s) = \varphi(x; s) \prod_{i=1}^n (1 - u s_i). \quad (2.10)$$

Note that  $E^{(x;s)}(u/s_1)$  acts naturally both on  $\mathbb{C}[[s^{-Q_+}]] [[x^{-Q_+}]]$  and on  $\mathbb{C}(s^{-Q_+}) [[x^{-Q_+}]]$ , with the coefficients  $A_I(x)$  regarded as elements of  $\mathbb{C}[[x^{-Q_+}]]$ .

**2.2. Proof of the unique existence.** In the following arguments, it is essential to transform this equation once more, into the eigenfunction equation for  $\psi(x; s)$  defined by

$$\varphi(x; s) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \psi(x; s), \quad \psi(x; s) = \sum_{\mu \in Q_+} x^{-\mu} \psi_\mu(s). \quad (2.11)$$

Note here that  $\varphi(x; s)$  and  $\psi(x; s)$  have a common leading coefficient; namely,  $\varphi_0(s) = \psi_0(s)$ . We define the  $q$ -difference operator  $L^{(x;s)}$  by setting

$$L^{(x;s)}(u) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} E^{(x;s)}(u) \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty}. \quad (2.12)$$

By this conjugation, we obtain the  $q$ -difference operator

$$\begin{aligned} L^{(x;s)}(u) &= \sum_{I \subseteq \{1, \dots, n\}} (-u)^{|I|} s^{\epsilon_I} B_I(x) T_{q,x}^{\epsilon_I}, \\ B_I(x) &= \prod_{i < j; i \notin I, j \in I} \frac{1 - tx_j/x_i}{1 - x_j/x_i} \frac{1 - qx_j/tx_i}{1 - qx_j/x_i} \quad (I \subseteq \{1, \dots, n\}). \end{aligned} \quad (2.13)$$

It should be observed that the coefficients  $B_I(x)$  of  $L^{(x;s)}(u)$  depend on  $t$  and  $q/t$ , symmetrically. We will investigate below the eigenfunction equation

$$L^{(x;s)}(u)\psi(x; s) = \psi(x; s) \prod_{i=1}^n (1 - us_i), \quad \psi(x; s) = \sum_{\nu \in Q_+} x^{-\nu} \psi_\nu(s) \quad (2.14)$$

for  $\psi(x; s)$ . Theorems 2.1 and 2.2 are derived from the following theorems concerning the eigenfunction equation (2.14).

**Theorem 2.3.** *For any  $c(s) \in \mathbb{C}[[s^{-Q_+}]]$  (resp.  $\in \mathbb{C}(s^{-Q_+})$ ), there exists a unique formal solution  $\psi(x; s)$  in  $\mathbb{C}[[s^{-Q_+}]][[x^{-Q_+}]]$  (resp. in  $\mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$ ) of (2.14) with leading coefficient  $\psi_0(s) = c(s)$ .*

**Theorem 2.4.** *Let  $\psi(x; s) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$  be the formal solution of (2.14) with  $\psi_0(s) = 1$ .*

(1) *For each  $\mu \in Q_+$ ,  $\psi_\mu(s) \in \mathbb{C}(s^{-Q_+})$  is regular at the origin  $(s_2/s_1, \dots, s_n/s_{n-1}) = 0$ , and has at most simple poles along the divisors  $q^{k+1}s_j/s_i = 1$  ( $1 \leq i < j \leq n$ ;  $k = 0, 1, 2, \dots$ ).*

(2) *This formal solution  $\psi(x; s)$  is invariant under the change of parameters  $t \leftrightarrow q/t$ .*

We remark that the gauge factor can be expanded as

$$\prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} = \sum_{\theta \in M_n} \prod_{1 \leq i < j \leq n} \frac{(t; q)_{\theta_{ij}}}{(q; q)_{\theta_{ij}}} (qx_j/tx_i)^{\theta_{ij}} \in \mathbb{C}[[x^{-Q_+}]] \quad (2.15)$$

by the  $q$ -binomial theorem, where  $M_n$  denotes the set of all strictly upper triangular  $n \times n$  matrices  $\theta = (\theta_{ij})_{i,j=1}^n$  with nonnegative integer coefficients:

$$M_n = \{ \theta = (\theta_{ij})_{i,j=1}^n \in \text{Mat}(n; \mathbb{N}) \mid \theta_{ij} = 0 \ (1 \leq j \leq i \leq n) \}. \quad (2.16)$$

Hence the coefficients  $\varphi_\mu(s)$  are expressed as  $\mathbb{C}$ -linear combinations of  $\psi_\nu(s)$ , and *vice versa*. Theorem 2.4, (1) for  $\psi(x; s)$  thus implies Theorem 2.2 for  $\varphi(x; s)$ . A characteristic feature of  $\psi(x; s)$  is the symmetry with respect to the change of parameters  $t \leftrightarrow q/t$ ; it is a consequence from the symmetry of the operator  $L^{(x;s)}(u)$  by the unique existence of a formal solution with leading coefficient 1. In the rest of this section, we give a proof of Theorem 2.3 and Theorem 2.4, (1).

By expanding the coefficients  $B_I(x)$  in the form

$$B_I(x) = \sum_{\mu \in Q_+} x^{-\mu} b_{\mu, I} \in \mathbb{C}[[x^{-Q_+}]], \quad (2.17)$$

we rewrite the operator  $L^{(x;s)}(u)$  as follows:

$$\begin{aligned} L^{(x;s)}(u) &= \sum_{\mu \in Q_+} x^{-\mu} b_{\mu}(u; s; T_{q,x}), \\ b_{\mu}(u; s; T_{q,x}) &= \sum_{I \subseteq \{1, \dots, n\}} (-u)^{|I|} b_{\mu, I} s^{\epsilon_I} T_{q,x}^{\epsilon_I} \quad (\mu \in Q_+). \end{aligned} \quad (2.18)$$

Note that the leading term of  $L^{(x;s)}(u)$  is given by

$$b_0(u; s; T_{q,x}) = \prod_{i=1}^n (1 - u s_i T_{q,x_i}), \quad (2.19)$$

since  $b_{0,I} = 1$  for all  $I \subseteq \{1, \dots, n\}$ , and that  $b_0(u; s; 1) = \prod_{i=1}^n (1 - u s_i)$  coincides with the eigenvalue in (2.14). By expanding the both sides of (2.14) as formal power series in the variables  $(x_2/x_1, \dots, x_n/x_{n-1})$ , we obtain the following recurrence relations for the coefficients  $\psi_{\mu}(u)$ :

$$(b_0(u; s; q^{-\mu}) - b_0(u; s; 1))\psi_{\mu}(s) + \sum_{0 < \nu \leq \mu} b_{\nu}(u; s; q^{-\mu+\nu})\psi_{\mu-\nu}(s) = 0 \quad (\mu \in Q_+), \quad (2.20)$$

where  $\leq$  denotes the dominance ordering in  $Q_+$ .

In order to show the unique existence of a formal solution with a given leading coefficient, we use the first order component

$$L_1^{(x;s)} = \sum_{j=1}^n s_j B_{\{j\}}(x) T_{q,x_j}, \quad B_{\{j\}}(x) = \prod_{i=1}^{j-1} \frac{(1 - t x_j/x_i)(1 - q x_j/t x_i)}{(1 - x_j/x_i)(1 - q x_j/x_i)}, \quad (2.21)$$

of  $L^{(x;s)}(u) = \sum_{r=0}^n (-u)^r L_r^{(x;s)}$ . Note that  $B_{\{j\}}(x) \in \mathbb{C}[[x_2/x_1, \dots, x_j/x_{j-1}]]$ , and hence  $b_{\mu, \{j\}} = 0$  unless  $\mu \in \mathbb{N}\alpha_1 + \dots + \mathbb{N}\alpha_{j-1}$ . The eigenfunction equation for  $\psi(x; s)$  now takes the form

$$L_1^{(x;s)}\psi(x; s) = \psi(x; s) \left( \sum_{j=1}^n s_j \right). \quad (2.22)$$

The corresponding recurrence relations for the coefficients  $\psi_{\mu}(s)$  are given by

$$(b_{0,1}(s; q^{-\mu}) - b_{0,1}(s; 1))\psi_{\mu}(s) + \sum_{0 < \nu \leq \mu} b_{\nu,1}(s; q^{-\mu+\nu})\psi_{\mu-\nu}(s) = 0 \quad (\mu \in Q_+), \quad (2.23)$$

where

$$\begin{aligned} b_{0,1}(s; q^{-\mu}) - b_{0,1}(s; 1) &= \sum_{j=1}^n s_j (q^{-\mu_j} - 1), \\ b_{\nu,1}(s; q^{-\mu+\nu}) &= \sum_{j=1}^n b_{\nu, \{j\}} s_j q^{-\mu_j + \nu_j} \quad (0 < \nu \leq \mu). \end{aligned} \quad (2.24)$$

Noting that  $b_{0,1}(s; q^{-\mu}) - b_{0,1}(s; 1) \neq 0$  for any  $\mu > 0$  as a polynomial in  $s = (s_1, \dots, s_n)$ , we immediately see that there exists a unique formal solution  $\psi(x; s) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$  of (2.22) with a given leading coefficient  $\psi_0(s) = c(s) \in \mathbb{C}(s^{-Q_+})$ .

Supposing that  $\mu > 0$ , choose the index  $r$  such that

$$\mu = k_r \alpha_r + \cdots + k_{n-1} \alpha_{n-1}, \quad k_r > 0. \quad (2.25)$$

This condition for  $r$  is equivalent to saying that  $\mu_i = 0$  ( $i < r$ ) and  $\mu_r \neq 0$ . Note that  $\mu_r = k_r$ ,  $\mu_j = -k_{j-1} + k_j$  ( $r < j < n$ ) and  $\mu_n = -k_{n-1}$ . In this case, for any  $\nu \in Q_+$  with  $0 < \nu \leq \mu$ , we have  $\nu \in \mathbb{N}\alpha_r + \cdots + \mathbb{N}\alpha_{n-1}$ . As we remarked already,  $b_{\nu, \{j\}} = 0$  unless  $\nu \in \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_{j-1}$ . This implies,  $b_{\nu, \{j\}} = 0$  for  $j = 1, \dots, r$ . Hence we have

$$\begin{aligned} b_{0,1}(s; q^{-\mu}) - b_{0,1}(s; 1) &= s_r(q^{-\mu_r} - 1) + \sum_{j=r+1}^n s_j(q^{-\mu_j} - 1), \\ b_{\nu,1}(s; q^{-\mu+\nu}) &= \sum_{j=r+1}^n b_{\nu, \{j\}} s_j q^{-\mu_j+\nu_j} \quad (0 < \nu \leq \mu), \end{aligned} \quad (2.26)$$

and the recurrence relation for  $\psi_\mu(s)$  is expressed as

$$((q^{-\mu_r} - 1) + \sum_{j>r} (q^{-\mu_j} - 1) s_j / s_r) \psi_\mu(s) + \sum_{0 < \nu \leq \mu} \left( \sum_{j>r} b_{\nu, \{j\}} q^{-\mu_j+\nu_j} s_j / s_r \right) \psi_{\mu-\nu}(s) = 0. \quad (2.27)$$

Since the coefficient of  $\psi_\mu(s)$  is invertible in  $\mathbb{C}[[s^{-Q_+}]]$  for any  $\mu > 0$ , this recurrence relation implies the unique existence of a formal solution  $\psi(x; s) \in \mathbb{C}[[s^{-Q_+}]][[x^{-Q_+}]]$  of (2.22) with a given leading coefficient  $\psi_0(s) = c(s) \in \mathbb{C}[[s^{-Q_+}]]$ .

This formal solution  $\psi(x; s)$  of (2.22), either in  $\mathbb{C}[[s^{-Q_+}]][[x^{-Q_+}]]$  or in  $\mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$ , actually solves the joint eigenfunction equation (2.14) with respect to the operator  $L^{(x;s)}(u)$ . Since the operator  $L^{(x;s)}(u)$  commute with  $L_1^{(x;s)}$ , the formal power series  $(L^{(x;s)}(u/s_1) - \prod_{i=1}^n (1 - u s_i / s_1)) \psi(x; s)$  is again a formal solution of (2.22). However, it must be zero by the uniqueness theorem since its leading coefficient is 0. This completes the proof of Theorem 2.3.

**2.3. Regularity of the expansion coefficients.** We now proceed to the proof of Theorem 2.4, (1). Let  $\psi(x; s) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$  be the formal solution of (2.14) with  $\psi_0(s) = 1$ . Then by induction on the dominance ordering, the recurrence relations

$$\psi_\mu(s) = \sum_{0 < \nu \leq \mu} \frac{\sum_{j>r} b_{\nu, \{j\}} q^{-\mu_j+\nu_j} s_j / s_r}{(1 - q^{-\mu_r}) + \sum_{j>r} (1 - q^{-\mu_j}) s_j / s_r} \psi_{\mu-\nu}(s) \quad (\mu > 0) \quad (2.28)$$

imply that  $\psi_\mu(s)$  is regular at  $(s_2/s_1, \dots, s_n/s_{n-1}) = 0$ . In fact, by these recurrence relations we can say more about the coefficients  $\psi_\mu(s)$ .

**Lemma 2.5.** *Let  $\psi(x; s) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$  be the formal solution of (2.14) with  $\psi_0(s) = 1$ . Suppose that  $\mu \in Q_+$  is expressed as  $\mu = k_r \alpha_r + \cdots + k_{n-1} \alpha_{n-1}$  for some  $r = 1, \dots, n$ :*

$$\mu \in \mathbb{N}\alpha_r + \cdots + \mathbb{N}\alpha_{n-1} \quad (r = 1, \dots, n). \quad (2.29)$$

*Then the coefficient  $\psi_\nu(s)$  is a rational function in  $(s_{r+1}/s_r, \dots, s_n/s_{n-1})$  regular at the origin  $(s_{r+1}/s_r, \dots, s_n/s_{n-1}) = 0$ :*

$$\psi_\mu(s) \in \mathbb{C}(s_{r+1}/s_r, \dots, s_n/s_{n-1}) \cap \mathbb{C}[[s_{r+1}/s_r, \dots, s_n/s_{n-1}]]. \quad (2.30)$$

□

What remains is to prove that the coefficients  $\psi_\mu(s) \in \mathbb{C}(s^{-Q_+})$  have at most simple poles along the divisors  $q^{k+1}s_j/s_i = 1$  ( $1 \leq i < j \leq n; k = 0, 1, 2, \dots$ ). Since  $\psi(x; s)$  is a formal solution of the joint eigenfunction equation (2.14), its coefficients satisfy the recurrence relations (2.20) as well:

$$(b_0(u; s; q^{-\mu}) - b_0(u; s; 1))\psi_\mu(s) + \sum_{0 < \nu \leq \mu} b_\nu(u; s; q^{-\mu+\nu})\psi_{\mu-\nu}(s) = 0 \quad (\mu \in Q_+). \quad (2.31)$$

Suppose that  $\mu \in Q_+$  is expressed as  $\mu = k_r\alpha_r + \dots + k_{n-1}\alpha_{n-1}$  with  $k_r > 0$ . In view of this recurrence relation, we investigate  $b_\nu(u; s; \xi)$ ,  $\xi = (\xi_1, \dots, \xi_n)$ , for  $\nu \in \mathbb{N}\alpha_r + \dots + \mathbb{N}\alpha_{n-1}$ .

**Lemma 2.6.** *For each  $\nu \in \mathbb{N}\alpha_r + \dots + \mathbb{N}\alpha_{n-1}$ , the polynomial  $b_\nu(u; s; \xi)$  is expressed as*

$$b_\nu(u; s; \xi) = \prod_{i=1}^{r-1} (1 - us_i\xi_i) b_\nu^{(r)}(u; s; \xi), \quad (2.32)$$

where  $b_\nu^{(r)}(u; s; \xi) = b_\nu(u; s_r, \dots, s_n; \xi_r, \dots, \xi_n)$  stands for the corresponding polynomial in the case of  $(n - r + 1)$  variables  $(x_r, \dots, x_n)$  and  $(s_r, \dots, s_n)$ .

*Proof.* Recall that these polynomials are determined by the expansion

$$\sum_{I \subseteq \{1, \dots, n\}} (-u)^{|I|} s^{\epsilon_I} B_I(x) \xi^{\epsilon_I} = \sum_{\nu \in Q_+} x^{-\nu} b_\nu(u; s; \xi). \quad (2.33)$$

In order to pick up  $b_\nu(u; s; \xi)$  for  $\nu \in \mathbb{N}\alpha_r + \dots + \mathbb{N}\alpha_{n-1}$ , we specialize the  $x$  variables by setting  $x_2/x_1 = \dots = x_r/x_{r-1} = 0$ . Since  $x_j/x_i = 0$  if  $i < j$  and  $i < r$ , for each  $I \subseteq \{1, \dots, n\}$  we have

$$\begin{aligned} B_I(x) \Big|_{x_{i+1}/x_i=0 \ (i=1, \dots, r-1)} &= \prod_{r \leq i < j \leq n; i \notin I, j \in I} \frac{1 - tx_j/x_i}{1 - x_j/x_i} \frac{1 - qx_j/tx_i}{1 - qx_j/x_i} \\ &= B_{I \cap \{r, \dots, n\}}(x_r, \dots, x_n), \end{aligned} \quad (2.34)$$

which depends only on  $I \cap \{r, \dots, n\}$ . Hence, for each  $\nu \in \mathbb{N}\alpha_r + \dots + \mathbb{N}\alpha_{n-1}$ , the polynomial  $b_\nu(u; s; \xi)$  can be expressed as

$$b_\nu(u; s; \xi) = \sum_{I \subseteq \{1, \dots, r-1\}} \sum_{J \subseteq \{r, \dots, n\}} (-u)^{|I \cup J|} b_{\nu, J}^{(r)} s^{\epsilon_{I \cup J}} \xi^{\epsilon_{I \cup J}} = \prod_{i=1}^{r-1} (1 - us_i\xi_i) b_\nu^{(r)}(u; s; \xi) \quad (2.35)$$

by means of the expansion coefficients  $b_{\nu, J}^{(r)}$  in the case of  $(n - r + 1)$  variables. □

In the setting of (2.31), by this lemma we have

$$\begin{aligned} b_0(u; s; q^{-\mu}) - b_0(u; s; 1) &= \prod_{i=1}^{r-1} (1 - us_i) \left( \prod_{j=r}^n (1 - us_j q^{-\mu_j}) - \prod_{j=r}^n (1 - us_j) \right), \\ b_\nu(u; s; q^{-\mu+\nu}) &= \prod_{i=1}^{r-1} (1 - us_i) b_\nu^{(r)}(u; s; q^{-\mu+\nu}). \end{aligned} \quad (2.36)$$

Hence we obtain the recurrence relation

$$\left(\prod_{j=r}^n (1 - us_j q^{-\mu_j}) - \prod_{j=r}^n (1 - us_j)\right) \psi_\mu(s) + \sum_{0 < \nu \leq \mu} b_\nu^{(r)}(u; s; q^{-\mu+\nu}) \psi_{\mu-\nu}(s) = 0. \quad (2.37)$$

Now by setting  $u = q^{k_r}/s_r = q^{\mu_r}/s_r$ , we can determine  $\psi_\mu(s)$  as follows:

$$\prod_{j=r}^n (1 - q^{k_r} s_j / s_r) \psi_\mu(s) = \sum_{0 < \nu \leq \mu} b_\nu^{(r)}(q^{k_r}/s_r; s; q^{-\mu+\nu}) \psi_{\mu-\nu}(s), \quad (2.38)$$

namely,

$$\psi_\mu(s) = \sum_{0 < \nu \leq \mu} \frac{b_\nu^{(r)}(q^{k_r}/s_r; s; q^{-\mu+\nu})}{(1 - q^{k_r}) \prod_{j=r+1}^n (1 - q^{k_r} s_j / s_r)} \psi_{\mu-\nu}(s). \quad (2.39)$$

Suppose that  $\nu_r = 0$ , or equivalently  $\nu \in \mathbb{N}\alpha_{r+1} + \cdots + \mathbb{N}\alpha_{n-1}$ . Then by Lemma 2.6 we have

$$b_\nu^{(r)}(u; s; q^{-\mu+\nu}) = (1 - us_r q^{-k_r}) b^{(r+1)}(u; s; q^{-\mu+\nu}), \quad (2.40)$$

and hence  $b_\nu^{(r)}(q^{k_r}/s_r; s; q^{-\mu+\nu}) = 0$  by the substitution  $u = q^{k_r}/s_r$ . This means that only such  $\nu \in Q_+$  that have positive coefficients on  $\alpha_r$  contribute nontrivially in the recurrence relation (2.39). In other words, we have the recurrence relation

$$\psi_\mu(s) = \sum_{0 \leq \nu < \mu; l_r < k_r} \frac{b_{\mu-\nu}^{(r)}(q^{k_r}/s_r; s; q^{-\nu})}{(1 - q^{k_r}) \prod_{j=r+1}^n (1 - q^{k_r} s_j / s_r)} \psi_\nu(s) \quad (2.41)$$

for each  $\mu \in Q_+$ ,  $\mu > 0$ , where  $k_r$  and  $l_r$  stands for the coefficient of  $\alpha_r$  in the expressions  $\mu = k_r \alpha_r + \cdots + k_{n-1} \alpha_{n-1}$  ( $k > 0$ ) and  $\nu = l_r \alpha_r + \cdots + l_{n-1} \alpha_{n-1}$ , respectively. By induction on the dominance ordering, we obtain the following.

**Proposition 2.7.** *Let  $\psi(x; s) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]]$  be the formal solution of (2.14) with  $\psi_0(s) = 1$ . Suppose that  $\mu \in Q_+$  is expressed as  $\mu = k_r \alpha_r + \cdots + k_{n-1} \alpha_{n-1}$  ( $r = 1, \dots, n$ ). Then the coefficient  $\psi_\mu(s)$  is expressed as*

$$\psi_\mu(s) = \frac{a_\nu(s)}{\prod_{r \leq i < j \leq n} (qs_j/s_i; q)_{k_i}} \quad (2.42)$$

for some polynomial  $a_\mu(s)$  in the variables  $(s_{r+1}/s_r, \dots, s_n/s_{n-1})$ .

□

This proposition is in fact a refinement of Theorem 2.4, (1).

**Remark 2.8.** Let us denote by  $\psi_n(x; s|q, t)$  the unique formal solution of (2.14) with leading coefficient 1. Express the eigenfunction equation (2.14) in the form

$$\sum_{I \subset \{1, \dots, n\}} (-u)^{|I|} s^{\epsilon_I} B_I(x) \psi_n(q^{\epsilon_I} x; s|q, t) = \psi_n(x; s|q, t) \prod_{i=1}^n (1 - us_i). \quad (2.43)$$

and specialize this formula by  $x_2/x_1 = \cdots = x_r/x_{r-1} = 0$ . From (2.34) it turns out that the formal power series  $\psi_n(x; s|q, t)|_{x_{i+1}/x_i=0 \ (i=1, \dots, r-1)}$  in  $(x_r/x_{r-1}, \dots, x_n/x_{n-1})$  satisfy

the eigenfunction equation (2.14) in  $(n - r + 1)$  variables  $(x_r, \dots, x_n)$  and  $(s_r, \dots, s_n)$ . Hence by the uniqueness of formal solution with a given leading coefficient, we have

$$\psi_n(x; s|q, t)|_{x_{i+1}/x_i=0 \ (i=1, \dots, r-1)} = \psi_{n-r+1}(x_r, \dots, x_n; s_r, \dots, s_n|q, t). \quad (2.44)$$

### 3. EXPLICIT FORMULAS FOR FORMAL EIGENFUNCTIONS

In this section we construct an explicit formal solution of the joint eigenfunction equation (2.1) in the form

$$f(x; s) = x^\lambda \varphi(x; s), \quad \varphi(x; s) \in \mathbb{C}[[s^{-Q_+}]][[x^{-Q_+}]] \quad (3.1)$$

with the complex parameters  $\lambda$  such that  $s = t^\delta q^\lambda$ .

**3.1. Formal power series  $p_n(x; s|q, t)$  and  $\varphi_n(x; s|q, t)$ .** We recall some notations in [10]. For each  $n = 1, 2, \dots$ , we denote by  $M_n$  the set of all strictly upper triangular  $n \times n$  matrices with nonnegative integer coefficients:

$$M_n = \{ \theta = (\theta_{ij})_{i,j=1}^n \in \text{Mat}(n; \mathbb{N}) \mid \theta_{ij} = 0 \ (1 \leq j \leq i \leq n) \}. \quad (3.2)$$

For each  $\theta \in M_n$ , we define a rational function  $c_n(\theta; s|q, t)$  in the variables  $s = (s_1, \dots, s_n)$  by

$$\begin{aligned} c_n(\theta; s|q, t) = & \prod_{k=2}^n \prod_{1 \leq i < j \leq k} \frac{(q^{\sum_{a>k} (\theta_{i,a} - \theta_{j,a})} t s_j / s_i; q)_{\theta_{i,k}}}{(q^{\sum_{a>k} (\theta_{i,a} - \theta_{j,a})} q s_j / s_i; q)_{\theta_{i,k}}} \\ & \cdot \prod_{k=2}^n \prod_{1 \leq i \leq j < k} \frac{(q^{-\theta_{j,k} + \sum_{a>k} (\theta_{i,a} - \theta_{j,a})} q s_j / t s_i; q)_{\theta_{i,k}}}{(q^{-\theta_{j,k} + \sum_{a>k} (\theta_{i,a} - \theta_{j,a})} s_j / s_i; q)_{\theta_{i,k}}}, \end{aligned} \quad (3.3)$$

where  $\sum_{a>k} = \sum_{a=k+1}^n$ . Note that, for each  $\theta \in M_n$ ,  $c_n(\theta; s|q, t)$  is in fact a rational function in  $(s_2/s_1, \dots, s_n/s_{n-1})$ , manifestly regular at the origin  $(s_2/s_1, \dots, s_n/s_{n-1}) = 0$ ; it can be regarded as an element of  $\mathbb{C}(s^{-Q_+}) \cap \mathbb{C}[[s^{-Q_+}]]$ . Note also that  $c_n(\theta; s|q, t)$  has at most (possibly multiple) poles along the divisors  $s_j/s_i = q^k$  ( $1 \leq i < j \leq n$ ;  $k \in \mathbb{Z}$ ).

These functions  $c_n(\theta; s|q, t)$  can be determined inductively on  $n$  starting from  $c_1(0; s_1|q, t) = 1$ . Let us parametrize  $\tilde{\theta} \in M_{n+1}$  by a pair  $(\theta, \nu) \in M_n \times \mathbb{N}^n$  as follows:

$$\tilde{\theta}_{i,j} = \theta_{i,j} \quad (1 \leq i < j \leq n), \quad \tilde{\theta}_{i,n+1} = \nu_i \quad (i = 1, \dots, n). \quad (3.4)$$

Then,  $c_{n+1}(\tilde{\theta}; s|q, t)$  is expressed as

$$c_{n+1}(\tilde{\theta}; s|q, t) = \prod_{1 \leq i < j \leq n+1} \frac{(t s_j / s_i; q)_{\nu_i}}{(q s_j / s_i; q)_{\nu_i}} \prod_{1 \leq i \leq j \leq n} \frac{(q^{-\nu_j} q s_j / t s_i; q)_{\nu_i}}{(q^{-\nu_j} s_j / s_i; q)_{\nu_i}} c_n(\theta; q^{-\nu} s|q, t). \quad (3.5)$$

**Example 3.1.** The rational functions  $c_n(\theta; s|q, t)$  for  $n = 2$  are given explicitly by

$$c_2(\theta_{12}; s_1, s_2|q, t) = \frac{(t s_2 / s_1; q)_{\theta_{12}}}{(q s_2 / s_1; q)_{\theta_{12}}} \frac{(q^{-\theta_{12}} q / t)_{\theta_{12}}}{(q^{-\theta_{12}})_{\theta_{12}}} = \frac{(t s_2 / s_1; q)_{\theta_{12}}}{(q s_2 / s_1; q)_{\theta_{12}}} \frac{(t; q)_{\theta_{12}}}{(q; q)_{\theta_{12}}} (q/t)^{\theta_{12}}. \quad (3.6)$$

For  $n = 3$ ,

$$c_3(\theta_{12}, \theta_{13}, \theta_{23}; s_1, s_2, s_3 | q, t) = \frac{(q^{\theta_{13}-\theta_{23}} t s_2 / s_1; q)_{\theta_{12}} (q^{-\theta_{12}} q / t; q)_{\theta_{12}}}{(q^{\theta_{13}-\theta_{23}} q s_2 / s_1; q)_{\theta_{12}} (q^{-\theta_{12}} q; q)_{\theta_{12}}} \cdot \frac{(t s_2 / s_1; q)_{\theta_{13}} (t s_3 / s_1; q)_{\theta_{13}} (t s_3 / s_2; q)_{\theta_{23}}}{(q s_2 / s_1; q)_{\theta_{13}} (q s_3 / s_1; q)_{\theta_{13}} (q s_3 / s_2; q)_{\theta_{23}}} \cdot \frac{(q^{-\theta_{13}} q / t; q)_{\theta_{13}} (q^{-\theta_{23}} q s_2 / t s_1; q)_{\theta_{13}} (q^{-\theta_{23}} q / t; q)_{\theta_{23}}}{(q^{-\theta_{13}} q; q)_{\theta_{13}} (q^{-\theta_{23}} q s_2 / s_1; q)_{\theta_{13}} (q^{-\theta_{23}} q; q)_{\theta_{23}}}. \quad (3.7)$$

For each  $n = 1, 2, \dots$ , we introduce a formal power series

$$p_n(x; s | q, t) = \sum_{\theta \in M_n} c_n(\theta; s | q, t) \prod_{1 \leq i < j \leq n} (x_j / x_i)^{\theta_{ij}} \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]] \quad (3.8)$$

with leading coefficient 1. For each  $\theta \in M_n$ , define an element  $\mu(\theta) \in Q_+$  by the formula

$$\prod_{1 \leq i < j \leq n} (x_j / x_i)^{\theta_{ij}} = x^{-\mu(\theta)}. \quad \mu(\theta) = \sum_{1 \leq i < j \leq n} \theta_{i,j} (\epsilon_i - \epsilon_j) \in Q_+. \quad (3.9)$$

For each  $\mu \in Q_+$ , we denote by  $M_n(\mu)$  the subset of  $M_n$  consisting of all  $\theta \in M_n$  such that  $\mu(\theta) = \mu$ ; this set  $M_n(\mu)$  is a finite set for each  $\mu \in Q_+$ . Then the power series expansion of  $p_n(x; s | q, t)$  is given by

$$p_n(x; s | q, t) = \sum_{\mu \in Q_+} x^{-\mu} p_\mu(s), \quad p_\mu(s) = \sum_{\theta \in M_n(\mu)} c_n(\theta; s | q, t) \quad (\mu \in Q_+). \quad (3.10)$$

We also remark that the recurrence relation (3.5) gives rise to a recurrence formula for  $p_n(x; s | q, t)$ :

$$p_{n+1}(x; s | q, t) = \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n+1} \frac{(t s_j / s_i; q)_{\nu_i}}{(q s_j / s_i; q)_{\nu_i}} \prod_{1 \leq i < j \leq n} \frac{(q^{-\nu_j+1} s_j / t s_i; q)_{\nu_i}}{(q^{-\nu_j} s_j / s_i; q)_{\nu_i}} \cdot \prod_{i=1}^n (q x_{n+1} / t x_i)^{\nu_i} p_n(x; q^{-\nu} s | q, t), \quad (3.11)$$

where we simply wrote  $p_{n+1}(x; s | q, t)$  in place of  $p_{n+1}(x_1, \dots, x_{n+1}; s_1, \dots, s_{n+1} | q, t)$  by using the same symbols  $x, s$  for  $(n+1)$  variables.

We introduce another formal power series  $\varphi_n(x; s | q, t) \in \mathbb{C}[[s^{-Q_+}]] [[x^{-Q_+}]]$  with a different normalization:

$$\begin{aligned} \varphi_n(x; s | q, t) &= \prod_{1 \leq i < j \leq n} \frac{(q s_j / s_i; q)_\infty}{(q s_j / t s_i; q)_\infty} p_n(x; s | q, t) \\ &= \prod_{1 \leq i < j \leq n} \frac{(q s_j / s_i; q)_\infty}{(q s_j / t s_i; q)_\infty} \sum_{\theta \in M_n} c_n(\theta; s | q, t) \prod_{1 \leq i < j \leq n} (x_j / x_i)^{\theta_{ij}}. \end{aligned} \quad (3.12)$$

Note that  $\varphi_1(x; s | q, t) = 1$ . For  $n = 2$ ,  $\varphi_2(x; s | q, t)$  is expressed in terms of  $q$ -hypergeometric series:

$$\begin{aligned} \varphi_2(x_1, x_2; s_1, s_2 | q, t) &= \frac{(q s_2 / s_1; q)_\infty}{(q s_2 / t s_1; q)_\infty} \sum_{k=0}^{\infty} \frac{(t s_2 / s_1; q)_k}{(q s_2 / s_1; q)_k} \frac{(t; q)_k}{(q; q)_k} (q x_2 / t x_1)^k \\ &= \frac{(q s_2 / s_1; q)_\infty}{(q s_2 / t s_1; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} t, t s_2 / s_1 \\ q s_2 / s_1 \end{matrix}; q, q x_2 / t x_1 \right]. \end{aligned} \quad (3.13)$$



By the definition (3.12), the leading coefficient of  $\varphi_n(x; s|q, t)$  in  $x$  variables is

$$\varphi_n(x; s|q, t) \Big|_{x_{i+1}/x_i=0 \ (i=1, \dots, n-1)} = \prod_{1 \leq i < j \leq n} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty}. \quad (3.14)$$

On the other hand, the leading coefficient as a formal power series in  $s$  variables is determined by using the recurrence formula (3.11) and the  $q$ -binomial theorem:

$$\varphi_n(x; s|q, t) \Big|_{s_{i+1}/s_i=0 \ (i=1, \dots, n-1)} = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty}. \quad (3.15)$$

**3.2. Relation to Macdonald polynomials.** The definition of the power series  $p_n(x; s|q, t)$  and  $\varphi_n(x; s|q, t)$  originates from the tableau representation (or the restriction formula) of the Macdonald polynomials. Recall from [4] that the monic Macdonald polynomial  $P_\lambda(x|q, t)$  associated with a partition  $\lambda = (\lambda_1, \dots, \lambda_n)$  is expressed explicitly as

$$P_\lambda(x|q, t) = \sum_{\phi=\mu^{(0)} \subseteq \mu^{(1)} \subseteq \dots \subseteq \mu^{(n)}=\lambda} \prod_{k=1}^n \psi_{\mu^{(k)}/\mu^{(k-1)}}(q, t) x_k^{|\mu^{(k)}/\mu^{(k-1)}|} \quad (3.16)$$

summed over increasing sequences of partitions of  $n$  steps from  $\phi$  to  $\lambda$ , where  $\psi_{\lambda/\mu}(q, t)$  is defined for each skew partition  $\lambda/\mu$  ( $\mu \subseteq \lambda$ ) by

$$\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i < j \leq n} \frac{(q^{\mu_i - \lambda_j + 1} t^{j-i-1}; q)_{\lambda_i - \mu_i}}{(q^{\mu_i - \lambda_j} t^{j-i}; q)_{\lambda_i - \mu_i}} \prod_{1 \leq i \leq j \leq n-1} \frac{(q^{\mu_i - \mu_j} t^{j-i+1}; q)_{\lambda_i - \mu_i}}{(q^{\mu_i - \mu_j + 1} t^{j-i}; q)_{\lambda_i - \mu_i}}. \quad (3.17)$$

Note that  $\psi_{\lambda/\mu}(q, t) = 0$  unless  $\lambda/\mu$  is a horizontal strip, namely,  $\mu_i \geq \lambda_{i+1}$  for all  $i$ .

By interpreting the representation (3.16) in terms of column strict tableaux, we denote by  $\theta_{i,j} = \mu_i^{(j)} - \mu_i^{(j-1)}$  the number of  $j$ 's in the  $i$ -th row for  $1 \leq i < j \leq n$ . Then the partitions  $\mu^{(k)}$  ( $k = 1, \dots, n$ ) are parametrized by  $\theta = (\theta_{i,j})_{i,j=1}^n \in M_n$  as  $\mu_i^{(k)} = \lambda_i - \sum_{a>k} \theta_{i,a}$  for  $1 \leq i < k$ . Through this parametrization together with  $s_i = t^{n-i} q^{\lambda_i}$  ( $i = 1, \dots, n$ ), one can directly verify that

$$\prod_{k=1}^n \psi_{\mu^{(k)}/\mu^{(k-1)}}(q, t) = c_n(\theta; s|q, t), \quad \prod_{k=1}^n x_k^{|\mu^{(k)}/\mu^{(k-1)}|} = x^\lambda \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{ij}}. \quad (3.18)$$

We now take  $c_n(\theta; s|q, t)$  for an arbitrary  $\theta \in M_n$ , and consider the specialization  $s = t^\delta q^\lambda$  by a partition  $\lambda$ , assuming that  $t$  is generic. Note that the factor  $(q^{\sum_{a>k} (\theta_{i,a} - \theta_{j,a})} t s_j / s_i; q)_{\theta_{i,k}}$  in the definition (3.3) with  $j = i + 1$  reduces to  $(q^{\sum_{a>k} (\theta_{i,a} - \theta_{i+1,a}) - \lambda_i + \lambda_{i+1}}; q)_{\theta_{i,k}}$  under this specialization. Suppose that

$$\prod_{1 \leq i < k \leq n} (q^{\sum_{a>k} (\theta_{i,a} - \theta_{i+1,a}) - \lambda_i + \lambda_{i+1}}; q)_{\theta_{i,k}} \neq 0. \quad (3.19)$$

The factors with  $k = n$  then imply  $\prod_{1 \leq i < n} (q^{-\lambda_i + \lambda_{i+1}}; q)_{\theta_{i,n}} \neq 0$ , and hence  $0 \leq \theta_{i,n} \leq \lambda_i - \lambda_{i+1}$  ( $1 \leq i < n$ ) since  $\lambda_i \geq \lambda_{i+1}$ . Starting from  $k = n$ , by descending induction on  $k$ , we see

$$0 \leq \theta_{i,k} \leq \lambda_i - \lambda_{i+1} - \sum_{a>k} (\theta_{i,a} - \theta_{i+1,a}) = \mu_i^{(k)} - \mu_{i+1}^{(k)} \quad (1 \leq i < k), \quad (3.20)$$

which confines  $\theta$  in a finite subset of  $M_n$  so that the monomial  $x^\lambda \prod_{1 \leq i \leq n} (x_j/x_i)^{\theta_{i,j}}$  remains as a polynomial in  $x$ . This shows that  $x^\lambda p_n(x; t^\delta q^\lambda) = P_\lambda(x|q, t)$  for any partition  $\lambda = (\lambda_1, \dots, \lambda_n)$ . The normalization factor for  $\varphi_n(x; s|q, t)$  takes care of the reciprocal of the evaluation formula

$$\begin{aligned} P_\lambda(t^\delta|q, t) &= t^{\sum_i (i-1)\lambda_i} \prod_{1 \leq i < j \leq n} \frac{(t^{j-i+1}; q)_{\lambda_i - \lambda_j}}{(t^{j-i}; q)_{\lambda_i - \lambda_j}} \\ &= t^{(\delta, \lambda)} \prod_{i=1}^n \frac{(q/t; q)_\infty}{(q/t^i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty} \end{aligned} \quad (3.21)$$

**Theorem 3.2.** *Suppose that the parameter  $t$  is generic in the sense that  $t^k \notin q^\mathbb{Z}$  for  $k = 1, \dots, n-1$ . Then, for any partition  $\lambda = (\lambda_1, \dots, \lambda_n)$ , the formal power series  $x^\lambda p_n(x; s|q, t)$  and  $x^\lambda \varphi_n(x; s|q, t)$  specialized by  $s = t^\delta q^\lambda = (t^{n-1}q^{\lambda_1}, \dots, q^{\lambda_n})$  recover the Macdonald polynomial  $P_\lambda(x|q, t)$  and a constant multiple of the normalized Macdonald polynomial  $\tilde{P}_\lambda(x|q, t)$ , respectively:*

$$\begin{aligned} x^\lambda p_n(x; t^\delta q^\lambda|q, t) &= P_\lambda(x; |q, t), \\ x^\lambda \varphi_n(x; t^\delta q^\lambda|q, t) &= t^{(\delta, \lambda)} \prod_{i=1}^n \frac{(q/t; q)_\infty}{(q/t^i; q)_\infty} \tilde{P}_\lambda(x|q, t). \end{aligned} \quad (3.22)$$

□

**3.3. Eigenfunction equation in  $x$  variables.** With the complex variables  $\lambda$  such that  $s = t^\delta q^\lambda$ , we consider the eigenfunction equation

$$D^x(u) f(x; s) = f(x; s) \prod_{i=1}^n (1 - us_i), \quad f(x; s) = x^\lambda \varphi(x; s) \in x^\lambda \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]], \quad (3.23)$$

or equivalently,

$$E^x(u) \varphi(x; s) = \varphi(x; s) \prod_{i=1}^n (1 - us_i), \quad \varphi(x; s) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]] \quad (3.24)$$

with leading coefficient  $\varphi_0(s) = 1$ . This equation is equivalent to an infinite set of linear recurrence relations for the coefficients  $\varphi_\mu(s) \in \mathbb{C}(s^{-Q_+})$  of  $\varphi(x; s)$ . When we take  $\varphi(x; s) = p_n(x; s|q, t)$ , we already know these relations hold under the specialization  $s = t^\delta q^\lambda$  for any partition  $\lambda$  of length  $\leq n$ , since  $p_n(x; t^\delta q^\lambda|q, t)$  coincides with  $P_\lambda(x|q, t)$ . This means that each of the recurrence relations holds for arbitrary partitions  $\lambda$ , hence it holds as an identity of rational functions. Hence we have

**Theorem 3.3.** *For the complex parameters  $\lambda$  with  $s = t^\delta q^\lambda$ , the formal power series  $x^\lambda p_n(x; s|q, t)$  satisfies the eigenfunction equation*

$$D^x(u) x^\lambda p_n(x; s|q, t) = x^\lambda p_n(x; s|q, t) \prod_{i=1}^n (1 - us_i). \quad (3.25)$$

Since  $x^\lambda p_n(x; s|q, t)$  is a formal solution with leading coefficient 1, by Theorem 2.2 we know that, for each  $\mu \in Q_+$ , the rational function

$$p_\mu(s) = \sum_{\theta \in M_n(\mu)} c_n(\theta; s|q, t) \quad (3.26)$$

has at most simple poles along the divisors  $s_j/s_i = q^{-k-1}$  ( $i < j; k = 0, 1, \dots$ ). This means that the poles at  $s_j/s_i = q^k$  with  $k = 0, 1, 2, \dots$ , which are apparent in  $c_n(\theta; s|q; t)$ , cancel out after the summation over all  $\theta \in M_n(\mu)$ .

**3.4. Symmetry with respect to  $t \leftrightarrow q/t$ .** Let us define a formal power series  $\psi(x; s)$  by

$$p_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \psi(x; s), \quad \psi_0(s) = 1. \quad (3.27)$$

Then by Theorem 2.4, (2),

$$\psi(x; s) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{ij}} \quad (3.28)$$

is invariant under the change of parameters  $t \leftrightarrow q/t$ .

**Proposition 3.4.** *The formal solution  $p_n(x; s|q, t)$  of (2.10) with leading coefficient 1 satisfies the symmetry relation*

$$p_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} p_n(x; s|q, q/t) \quad (3.29)$$

with respect to the change of parameters  $t \leftrightarrow q/t$ . Namely, we have the transformation formula

$$\begin{aligned} & \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{ij}} \\ &= \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \sum_{\theta \in M_n} c_n(\theta; s|q, q/t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{ij}}. \end{aligned} \quad (3.30)$$

This formula for  $n = 2$

$${}_2\phi_1 \left[ \begin{matrix} t, & ts_2/s_1 \\ & qs_2/s_1 \end{matrix}; q, qx_2/tx_1 \right] = \frac{(tx_2/x_1)_\infty}{(qx_2/tx_1)_\infty} {}_2\phi_1 \left[ \begin{matrix} q/t, & qs_2/ts_1 \\ & qs_2/s_1 \end{matrix}; q, tx_2/x_1 \right] \quad (3.31)$$

is the  $q$ -Euler transformation formula for  ${}_2\phi_1$ . In terms of the formal solution  $\varphi_n(x; s|q, t)$ , the transformation formula mentioned above is expressed as

$$\varphi_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \frac{(ts_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} \varphi_n(x; s|q, q/t). \quad (3.32)$$

**Remark 3.5.** In this section, we made use of the explicit formula for Macdonald polynomials to prove that  $p_n(x; s|q, t)$  and  $\varphi_n(x; s|q, t)$  are formal solutions of the eigenfunction equation in the  $x$  variables. As we will see in the next section on, however, Theorem 3.3 as well as Proposition 3.4 can be proved without relying on the theory of Macdonald polynomials.

#### 4. RECURRENCE BY JACKSON INTEGRALS

In this section, we show that the explicit formal solution  $f(x; s) = x^\lambda \varphi_n(x; s|q, t)$  of the eigenfunction equation in  $x$  variables essentially solves the bispectral problem

$$D^x(u) f(x; s) = f(x; s) \prod_{i=1}^n (1 - us_i), \quad D^s(u) f(x; s) = f(x; s) \prod_{i=1}^n (1 - ux_i). \quad (4.1)$$

**4.1. Preliminary remarks.** Let  $e_n(x; s)$  be a (possibly multi-valued) meromorphic function in  $(x, s)$  such that

$$T_{q, x_i} e_n(x; s) = e_n(x; s) s_i t^{-n+i}, \quad T_{q, s_i} e_n(x; s) = e_n(x; s) x_i t^{-n+i} \quad (i = 1, \dots, n) \quad (4.2)$$

and that  $e_n(x; s) = e_n(s; x)$ . In order to fix the idea, we choose now the function

$$e_n(x; s) = q^{\sum_{i=1}^n \kappa_i \lambda_i}, \quad (4.3)$$

defined in terms of additive complex variables  $\kappa = (\kappa_1, \dots, \kappa_n)$  and  $\lambda = (\lambda_1, \dots, \lambda_n)$  such that  $x_i = t^{n-i} q^{\kappa_i}$ ,  $s_i = t^{n-i} q^{\lambda_i}$  ( $i = 1, \dots, n$ ). From the relation  $x = t^\delta q^\kappa$  and  $s = t^\delta q^\lambda$ , we have

$$e_n(x; s) = x^\lambda t^{-\langle \delta, \lambda \rangle} = s^\kappa t^{-\langle \delta, \kappa \rangle}. \quad (4.4)$$

We remark that the function

$$e_n(x; s) = \prod_{i=1}^n \frac{\theta(x_i t^{n-i}; q) \theta(s_i t^{n-i}; q)}{\theta(x_i s_i; q)}. \quad (4.5)$$

defined by theta functions is an alternative choice for  $e_n(x; s)$ .

We regard  $\varphi_n(x; s|q, t)$  as a formal power series in  $(x_2/x_1, \dots, x_n/x_{n-1}; s_2/s_1, \dots, s_n/s_{n-1})$ :

$$\varphi_n(x; s|q, t) = \sum_{\mu, \nu \in Q_+} x^{-\mu} s^{-\nu} \varphi_{\mu, \nu} \in \mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]]. \quad (4.6)$$

Our goal is to show that the formal power series

$$f_n(x; s|q, t) = e_n(x; s) \varphi_n(x; s|q, t) \in e_n(x; s) \mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]] \quad (4.7)$$

is a solution of the bispectral problem (4.1). Since  $e_n(x; s) = x^\lambda t^{-\langle \delta, \lambda \rangle}$ , we already know that  $f_n(x; s|q, t)$  is a solution of the eigenfunction equation in  $x$  variables. We need to show that this formal power series simultaneously solves the eigenfunction equation in  $s$  variables. For that purpose we will make use of the inductive construction of eigenfunctions by Jackson integrals.

Before investigating the general case, we observe the case  $n = 2$  in some detail. Recall that

$$\begin{aligned} \varphi_2(x; s|q, t) &= \frac{(qs_2/s_1; q)_\infty}{(qs_2/ts_1; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} t, ts_2/s_1 \\ qs_2/s_1 \end{matrix}; q, qx_2/tx_1 \right] \\ &= \frac{(qs_2/s_1; q)_\infty}{(qs_2/ts_1; q)_\infty} \sum_{k=0}^{\infty} \frac{(t; q)_k (ts_2/s_1; q)_k}{(qs_2/s_1; q)_k (q; q)_k} (qx_2/tx_1)^k. \end{aligned} \quad (4.8)$$

This series defines a holomorphic function on the domain  $|qx_2/tx_1| < 1$  and  $qs_2/ts_1 \notin q^{-\mathbb{N}}$ . Suppose that  $|t| < 1$ . Then by

$$\frac{(t; q)_k}{(qs_2/s_1; q)_k} = \frac{(t; q)_\infty (q^{k+1}s_2/s_1; q)_\infty}{(qs_2/s_1; q)_\infty (q^k t; q)_\infty} = \frac{(t; q)_\infty}{(qs_2/s_1; q)_\infty} \sum_{l=0}^{\infty} \frac{(qs_2/ts_1; q)_l}{(q; q)_l} q^{kl} t^l, \quad (4.9)$$

we have

$$\begin{aligned} \varphi_2(x; s|q, t) &= \frac{(t; q)_\infty}{(qs_2/ts_1; q)_\infty} \sum_{k,l} \frac{(qs_2/ts_1; q)_l}{(q; q)_l} \frac{(ts_2/s_1; q)_k}{(q; q)_k} (qx_2/tx_1)^k q^{kl} t^l \\ &= \frac{(t; q)_\infty}{(qs_2/ts_1; q)_\infty} \sum_l \frac{(qs_2/ts_1; q)_l}{(q; q)_l} \frac{(q^{l+1}x_2s_2/x_1s_1; q)_\infty}{(q^{l+1}x_2/tx_1; q)_\infty} t^l \\ &= \frac{(t; q)_\infty (qx_2s_2/x_1s_1; q)_\infty}{(qs_2/ts_1; q)_\infty (qx_2/tx_1; q)_\infty} \sum_l \frac{(qs_2/ts_1; q)_l (qx_2/tx_1; q)_l}{(q; q)_l (qx_2s_2/x_1s_1; q)_l} t^l. \end{aligned} \quad (4.10)$$

Hence

$$\varphi_2(x; s|q, t) = \frac{(t; q)_\infty (qx_2s_2/x_1s_1; q)_\infty}{(qx_2/tx_1; q)_\infty (qs_2/ts_1; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} qx_2/tx_1, qs_2/ts_1 \\ qx_2s_2/x_1s_1 \end{matrix}; q, t \right] \quad (|t| < 1). \quad (4.11)$$

This expression represents a meromorphic function in  $(x_2/x_1, s_2/s_1) \in \mathbb{C} \times \mathbb{C}$  at most with simple poles at  $qx_2/tx_1 \in q^{-\mathbb{N}}$  and  $qs_2/ts_1 \in q^{-\mathbb{N}}$ . Furthermore  $\varphi_2(x; s|q, t)$  is manifestly symmetric with respect to  $x$  and  $s$  variables. This immediately implies that the meromorphic function  $f_2(x; s|q, t) = e_2(x; s)\varphi_2(x; s|q, t)$  actually solves the bispectral problem. See [12, Proposition 5.4] also.

For general  $n$ , it is a challenging problem to find an analytic expression that directly implies symmetry between  $x$  and  $s$  variables; the case  $n = 3$  will be studied explicitly in Section 7. In the following, we will show that  $f_n(x; s|q, t)$  satisfies the eigenfunction equation for  $s$  variables by means of the Jackson integral representation. The symmetry with respect to  $x$  and  $s$  variables will be derived as a consequence of the fact that it satisfies the two eigenfunction equations simultaneously.

**4.2. Recurrence by Jackson integrals.** In order to prove that  $f_n(x; s|q, t) = e_n(x; s) \times \varphi_n(x; s|q, t)$  satisfies the eigenfunction equation in  $s$  variables, by exchanging the roles of  $x$  and  $s$  variables we show that

$$f_n(s; x|q, t) = e_n(s; x)\varphi_n(s; x|q, t) \quad (4.12)$$

satisfies the eigenfunction equation in the  $x$  variables. For the proof we make use of a result of [7] concerning the integral representation of eigenfunctions for Ruijsenaars-Macdonald operators. We remark that the method of [7] is applicable to our formal setting as well, since it is based on the  $q$ -difference equation satisfied by a kernel function.

Let  $g(y)$  be a joint eigenfunction of Ruijsenaars-Macdonald operators in  $n$  variables  $y = (y_1, \dots, y_n)$  such that

$$D^y(u)g(y) = g(y) \prod_{i=1}^n (1 - ut^{n-i}q^{\lambda_i}) \quad (4.13)$$

with complex parameters  $\lambda = (\lambda_1, \dots, \lambda_n)$ . Then, for  $m \geq n$ , we define a function  $f(x)$  in  $m$  variables  $x = (x_1, \dots, x_m)$  by an integral transformation of the form

$$f(x) = \int H_\kappa(x; y) g(y) d\omega(y) \quad (4.14)$$

with a complex parameter  $\kappa$ , where  $d\omega(y)$  is a  $q$ -invariant measure on  $\mathbb{T}_y^n$ . We choose the kernel function so that

$$H_\kappa(x; y) \equiv (x_1 \cdots x_m)^\kappa \prod_{i=1}^n \prod_{j=1}^m \frac{(tx_j/y_i; q)_\infty}{(x_j/y_i; q)_\infty} \prod_{1 \leq i, j \leq n; i \neq j} \frac{(y_i/y_j; q)_\infty}{(ty_i/y_j; q)_\infty} (y_1 \cdots y_n)^{-\kappa}, \quad (4.15)$$

where  $\equiv$  means that the both sides coincides up to multiplication by a *quasi-constant*, namely, by a  $q$ -periodic function in all variables  $x_j$  and  $y_i$ . Note that  $H_\kappa(x; y)$  is essentially the Cauchy kernel (with  $y$  variables reversed) multiplied by the weight function of orthogonality for Macdonald polynomials. We assume that the integral (4.14) makes sense and that the actions of  $T_{q, x_j}$  ( $j = 1, \dots, m$ ) commute with the integral. Then, it is known by [7] that  $f(x)$  is as well a joint eigenfunction in  $x$  variables and satisfies

$$D^x(u)f(x) = f(x) \prod_{j=1}^m (1 - ut^{m-j}q^{\lambda_j}), \quad (4.16)$$

with additional parameters  $\lambda_j = \kappa$  ( $j = n+1, \dots, m$ ).

We apply this integral transformation to formal power series, in the case where  $m = n+1$  and  $\kappa = \lambda_{n+1}$ . For that purpose, we rewrite (4.14) in terms of  $\psi(y)$  and  $\varphi(x)$  defined by

$$g(y) = y_1^{\lambda_1} \cdots y_n^{\lambda_n} \psi(y), \quad f(x) = x_1^{\lambda_1} \cdots x_{n+1}^{\lambda_{n+1}} \varphi(x), \quad (4.17)$$

respectively. Then  $\varphi(x)$  should be obtained from  $\psi(y)$  by integral transformation

$$\varphi(x) = \int K_\lambda(x; y) \psi(y) d\omega(y) \quad (4.18)$$

with a kernel such that

$$\begin{aligned} K_\lambda(x; y) &\equiv x_1^{-\lambda_1} \cdots x_n^{-\lambda_n} H_{\lambda_{n+1}}(x; y) y_1^{\lambda_1} \cdots y_n^{\lambda_n} \\ &= \prod_{i=1}^n \prod_{j=1}^{n+1} \frac{(tx_j/y_i; q)_\infty}{(x_j/y_i; q)_\infty} \prod_{1 \leq i, j \leq n; i \neq j} \frac{(y_i/y_j; q)_\infty}{(ty_i/y_j; q)_\infty} \prod_{i=1}^n (y_i/x_i)^{\lambda_i - \lambda_{n+1}}. \end{aligned} \quad (4.19)$$

Multiplying the right hand side by the quasi-constant

$$\prod_{1 \leq i < j \leq n} \frac{\theta(x_i/y_j; q)}{\theta(tx_i/y_j; q)} \frac{\theta(ty_i/y_j; q)}{\theta(y_i/y_j; q)} \prod_{i=1}^n (y_i/x_i)^{(n-i)\beta} (q/t)^{\lambda_i - \lambda_{n+1} + (n-i)\beta}, \quad t = q^\beta, \quad (4.20)$$

we choose the function

$$\begin{aligned} K_\lambda(x; y) &= \prod_{i=1}^n \frac{(tx_i/y_i; q)_\infty}{(x_i/y_i; q)_\infty} \prod_{1 \leq i < j \leq n+1} \frac{(tx_j/y_i; q)_\infty}{(x_j/y_i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(qy_j/x_i; q)_\infty}{(qy_j/tx_i; q)_\infty} \\ &\quad \cdot \prod_{1 \leq i < j \leq n} \frac{(y_j/y_i; q)_\infty}{(ty_j/y_i; q)_\infty} \frac{(qy_j/tx_i; q)_\infty}{(qy_j/y_i; q)_\infty} \prod_{i=1}^n (qy_i/tx_i)^{\lambda_i - \lambda_{n+1} + (n-i)\beta} \end{aligned} \quad (4.21)$$

with parameters  $\lambda = (\lambda_1, \dots, \lambda_{n+1})$  for our kernel; a power of  $q/t$  is introduced for the sake of normalization. Noting that this function has zeros at  $y_i = q^k tx_i$  ( $i = 1, \dots, n$ ;  $k = 0, 1, 2, \dots$ ), as the  $q$ -invariant measure we make use of the Jackson integral

$$\varphi(x) = \frac{1}{(1-q)^n} \int_{tx_1}^{\infty} \cdots \int_{tx_n}^{\infty} K_{\lambda}(x; y) \psi(y) \frac{d_q y_1}{y_1} \cdots \frac{d_q y_n}{y_n}. \quad (4.22)$$

Then, by the argument of [7] applied to formal power series, we can formulate the following inductive construction of formal eigenfunctions.

**Lemma 4.1.** *Let*

$$\psi(x; s) \in \mathbb{C}[[x^{-Q_+}]][[s^{-Q_+}]], \quad (x; s) = (x_1, \dots, x_n; s_1, \dots, s_n) \quad (4.23)$$

*be a formal power series such that  $g(x; s) = e_n(x; s)\psi(x; s)$  is a formal solution of the eigenfunction equation (2.1) in  $n$  variables  $x = (x_1, \dots, x_n)$ . Define a formal power series*

$$\varphi(x; s) \in \mathbb{C}[[x^{-Q_+}]][[s^{-Q_+}]], \quad (x; s) = (x_1, \dots, x_{n+1}; s_1, \dots, s_{n+1}), \quad (4.24)$$

*by the Jackson integral*

$$\varphi(x; s) = \frac{1}{(1-q)^n} \int_{tx_1}^{\infty} \cdots \int_{tx_n}^{\infty} K_{\lambda}(x; y) \psi(y; s) \frac{d_q y_1}{y_1} \cdots \frac{d_q y_n}{y_n}, \quad (4.25)$$

*where  $K_{\lambda}(x; y)$  is the kernel defined by (4.21) with complex parameters  $\lambda = (\lambda_1, \dots, \lambda_{n+1})$  such that  $s_i = t^{n+1-i} q^{\lambda_i}$  ( $i = 1, \dots, n+1$ ). Then  $f(x; s) = e_{n+1}(x; s)\varphi(x; s)$  is a formal solution of the eigenfunction equation in  $(n+1)$  variables  $x = (x_1, \dots, x_{n+1})$ .*

□

The Jackson integral (4.25) is in fact an  $n$ -tuple sum of the values of the integrand at the points

$$(y_1, \dots, y_n) = (q^{-\nu_1-1} tx_1, \dots, q^{-\nu_n-1} tx_n), \quad \nu = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n. \quad (4.26)$$

It is expressed as

$$\begin{aligned} \varphi(x; s) &= \sum_{\nu \in \mathbb{N}^n} K_{\lambda}(x; q^{-\nu} tx/q) \psi(q^{-\nu} tx/q; s) \\ &= \left( \frac{(q; q)_{\infty}}{(q/t; q)_{\infty}} \right)^n \prod_{1 \leq i < j \leq n+1} \frac{(qx_j/x_i; q)_{\infty}}{(qx_j/tx_i; q)_{\infty}} \\ &\quad \cdot \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(q/t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n+1} \frac{(qx_j/tx_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \prod_{1 \leq i < j \leq n} \frac{(q^{-\nu_j} tx_j/x_i; q)_{\nu_i}}{(q^{-\nu_j} x_j/x_i; q)_{\nu_i}} \\ &\quad \cdot \prod_{i=1}^n (ts_{n+1}/s_i)^{\nu_i} \prod_{1 \leq i < j \leq n} \frac{(q^{\nu_i-\nu_j+1} x_j/tx_i; q)_{\infty}}{(q^{\nu_i-\nu_j+1} x_j/x_i; q)_{\infty}} \psi(q^{-\nu} x; s), \end{aligned} \quad (4.27)$$

where  $\psi(q^{-\nu} tx/q; s)$  is replaced by  $\psi(q^{-\nu} x; s)$  by using the homogeneity of  $\psi(x; s)$ .

In view of Lemma 4.1, starting from  $\phi_1(s; x|q, t)=1$ , we define a sequence of formal power series

$$\phi_n(s; x|q, t) \in \mathbb{C}[[x^{-Q_+}]][[s^{-Q_+}]], \quad (x; s) = (x_1, \dots, x_n; s_1, \dots, s_n) \quad (n = 1, 2, \dots) \quad (4.28)$$

inductively by

$$\begin{aligned}
& \phi_{n+1}(s; x|q, t) \\
&= \prod_{1 \leq i < j \leq n+1} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \\
&\quad \cdot \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(q/t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n+1} \frac{(qx_j/tx_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \prod_{1 \leq i < j \leq n} \frac{(q^{-\nu_j} tx_j/x_i; q)_{\nu_i}}{(q^{-\nu_j} x_j/x_i; q)_{\nu_i}} \\
&\quad \cdot \prod_{i=1}^n (ts_{n+1}/s_i)^{\nu_i} \prod_{1 \leq i < j \leq n} \frac{(q^{\nu_i - \nu_j + 1} x_j/tx_i; q)_\infty}{(q^{\nu_i - \nu_j + 1} x_j/x_i; q)_\infty} \phi_n(s; q^{-\nu} x|q, t). \tag{4.29}
\end{aligned}$$

Then we obtain a sequence of formal solutions

$$e_n(x; s) \phi_n(s; x|q, t) \in e_n(x; s) \mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]] \tag{4.30}$$

of the eigenfunction equations (2.1) in  $x$  variables. By the  $q$ -binomial theorem, we can determine the leading coefficient of  $\phi_n(s; x|q, t)$  inductively as

$$\phi_n(s; x|q, t) \Big|_{x_{i+1}/x_i=0 \ (i=1, \dots, n-1)} = \prod_{1 \leq i < j \leq n} \frac{(qs_j/s_i; q)_\infty}{(ts_j/s_i; q)_\infty}. \tag{4.31}$$

In view of Theorem 2.4, (2), we introduce

$$\psi_n(s; x|q, t) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} \frac{(ts_j/s_i; q)_\infty}{(qs_j/s_i; q)_\infty} \phi_n(s; x|q, t) \quad (n = 1, 2, \dots) \tag{4.32}$$

so that  $\psi_n(s; x|q, t)$  has leading coefficient 1 in  $x$ , and becomes invariant under the change of parameters  $t \leftrightarrow q/t$ . We rewrite the recurrence formula (4.29) for  $\phi_n(s; x|q, t)$  into that for  $\psi_n(s; x|q, t)$ :

$$\begin{aligned}
\psi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(ts_{n+1}/s_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(q/t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n+1} \frac{(qx_j/tx_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \\
&\quad \cdot \prod_{1 \leq i < j \leq n} \frac{(q^{-\nu_j} tx_j/x_i; q)_{\nu_i}}{(q^{-\nu_j} x_j/x_i; q)_{\nu_i}} \prod_{i=1}^n (ts_{n+1}/s_i)^{\nu_i} \psi_n(s; q^{-\nu} x|q, t). \tag{4.33}
\end{aligned}$$

By the symmetry of  $\psi_n(s; x|q, t)$ , we can exchange  $t$  and  $q/t$  so that

$$\begin{aligned}
\psi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(qs_{n+1}/ts_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n+1} \frac{(tx_j/x_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \\
&\quad \cdot \prod_{1 \leq i < j \leq n} \frac{(q^{-\nu_j} qx_j/tx_i; q)_{\nu_i}}{(q^{-\nu_j} x_j/x_i; q)_{\nu_i}} \prod_{i=1}^n (qs_{n+1}/ts_i)^{\nu_i} \psi_n(s; q^{-\nu} x|q, t). \tag{4.34}
\end{aligned}$$

By comparing this with the recurrence formula (3.11) for  $p_n(x; s|q, t)$ , we conclude that

$$\psi_n(s; x|q, t) = \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty} p_n(s; x|q, t) \tag{4.35}$$



with the role of  $x$  and  $s$  variables exchanged, and hence

$$\begin{aligned}\phi_n(s; x|q, t) &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \frac{(qs_j/ts_i; q)_\infty}{(ts_j/s_i; q)_\infty} p_n(s; x|q, t) \\ &= \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_\infty}{(ts_j/s_i; q)_\infty} \varphi_n(s; x|q, t).\end{aligned}\quad (4.36)$$

This shows that  $e_n(x; s)\varphi_n(s; x|q, t)$  is a formal solution of the eigenfunction equation (2.1) in  $x$  variables. It also turns out that

$$\begin{aligned}\psi_n(s; x|q, t) &= \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty} p_n(s; x|q, t) \\ &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty} \varphi_n(s; x|q, t)\end{aligned}\quad (4.37)$$

is invariant under the change of parameters  $t \leftrightarrow q/t$ , which reproves the transformation formulas (3.29) and (3.32).

**Theorem 4.2.** *The sequence of formal power series  $\varphi_n(s; x|q, t)$  ( $n=1, 2, \dots$ ) satisfies the following recurrence relation by Jackson integral transformations:*

$$\begin{aligned}\varphi_{n+1}(s; x|q, t) &= \left( \frac{(q/t; q)_\infty}{(1-q)(q; q)_\infty} \right)^n \prod_{i=1}^n \frac{(ts_{n+1}/s_i; q)_\infty}{(qs_{n+1}/ts_i; q)_\infty} \\ &\quad \cdot \int_{tx_1}^\infty \cdots \int_{tx_n}^\infty K_\lambda(x; y) \varphi_n(s; y|q, t) \frac{d_q y_1}{y_1} \cdots \frac{d_q y_n}{y_n}\end{aligned}\quad (4.38)$$

where  $K_\lambda(x; y)$  is the kernel defined by (4.21).

□

This Jackson integral transformation corresponds to the following recurrence formula for  $\varphi_n(s; x|q, t)$ :

$$\begin{aligned}\varphi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(ts_{n+1}/s_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \prod_{1 \leq i < j \leq n+1} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \\ &\quad \cdot \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(q/t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n+1} \frac{(qx_j/tx_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \frac{(q^{-\nu_j} tx_j/x_i; q)_{\nu_i}}{(q^{-\nu_j} x_j/x_i; q)_{\nu_i}} \\ &\quad \cdot \prod_{i=1}^n (ts_{n+1}/s_i)^{\nu_i} \prod_{1 \leq i < j \leq n} \frac{(q^{\nu_i - \nu_j + 1} x_j/tx_i; q)_{\nu_i}}{(q^{\nu_i - \nu_j + 1} x_j/x_i; q)_{\nu_i}} \varphi_n(s; q^{-\nu} x).\end{aligned}\quad (4.39)$$

**Theorem 4.3.** *For each  $n = 1, 2, \dots$ , the formal power series*

$$\begin{aligned}f(x; s) &= x^\lambda \varphi_n(s; x|q, t) \\ &= x^\lambda \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} p_n(s; x|q, t) \in x^\lambda \mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]]\end{aligned}\quad (4.40)$$

defined with parameters  $\lambda$  such that  $s = t^\delta q^\lambda$  is a formal solution of the eigenfunction equation (2.1) in  $x$  variables.

□

**4.3. Formal solution of the bispectral problem.** In Theorem 3.3 and Theorem 4.3, we proved that  $x^\lambda \varphi_n(x; s|q, t)$  and  $x^\lambda \varphi_n(s; x|q, t)$  satisfy the eigenfunction equation in the  $x$  variables, respectively. As we already remarked in Section 2,  $\varphi_n(x; s|q, t)$  has the leading coefficient

$$\varphi_n(x; s|q, t) \Big|_{s_{i+1}/s_i=0 \ (i=1, \dots, n-1)} = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \quad (4.41)$$

in  $s$  variables. This means that the two formal solutions  $x^\lambda \varphi_n(x; s|q, t)$  and  $x^\lambda \varphi_n(s; x|q, t)$  of the eigenfunction equation in  $x$  variables have the same leading coefficient

$$\prod_{1 \leq i < j \leq n} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty}. \quad (4.42)$$

Hence by Theorem 2.1 we see  $\varphi_n(x, s|q, t)$  and  $\varphi_n(s, x|q, t)$  coincides as formal power series in  $\mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]]$ .

Summarizing these arguments, we have

**Theorem 4.4.** *Let  $e_n(x; s)$  be a solution of the  $q$ -difference equations (4.2) with symmetry  $e_n(x; s) = e_n(s; x)$ . Then*

$$f_n(x; s|q, t) = e_n(x; s) \varphi_n(x; s|q, t) \in e_n(x; s) \mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]] \quad (4.43)$$

*is a formal solution of the bispectral problem (4.1). The formal power series  $\varphi_n(x; s|q, t)$  have leading coefficients*

$$\begin{aligned} \varphi_n(x; s|q, t) \Big|_{x_{i+1}/x_i=0 \ (i=1, \dots, n-1)} &= \prod_{1 \leq i < j \leq n} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty}, \\ \varphi_n(x; s|q, t) \Big|_{s_{i+1}/s_i=0 \ (i=1, \dots, n-1)} &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty}, \end{aligned} \quad (4.44)$$

*each of which determines the formal solution uniquely. It has symmetry*

$$\varphi_n(x; s|q, t) = \varphi_n(s; x|q, t) \quad (4.45)$$

*between  $x$  and  $s$  variables, and transforms as*

$$\varphi_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \frac{(ts_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} \varphi_n(x; s|q, q/t). \quad (4.46)$$

*under the change of parameters  $t \leftrightarrow q/t$ .*

□

From the symmetry  $\varphi_n(x; s|q, t) = \varphi_n(s; x|q, t)$ , it turns out that the formal power series  $\psi_n(x; s|q, t)$  of (4.37) has symmetry  $\psi_n(x; s|q, t) = \psi_n(s; x|q, t)$  as well. Namely, the formal power series

$$\begin{aligned} \psi_n(x; s|q, t) &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} p_n(x; s|q, t) \\ &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{ij}} \end{aligned} \quad (4.47)$$

satisfies

$$\psi_n(x; s|q, t) = \psi_n(s; x|q, t) \quad \text{and} \quad \psi_n(x; s|q, t) = \psi_n(x; s|q, q/t). \quad (4.48)$$

We will prove later that this  $\psi_n(x; s|q, t)$  represents a meromorphic function on  $\mathbb{C}^{n-1} \times \mathbb{C}^{n-1}$  with coordinates  $(x_2/x_1, \dots, x_n/x_{n-1}; s_2/s_1, \dots, s_n/s_{n-1})$  at most with simple poles along

$$q^{k+1}x_j/x_i = 1, \quad q^{k+1}s_j/s_i = 1 \quad (1 \leq i < j \leq n; k = 0, 1, 2, \dots). \quad (4.49)$$

## 5. RECURRENCE BY $q$ -DIFFERENCE OPERATORS

In the previous section, we described the inductive structure for  $\varphi_n(x; s|q, t)$  ( $n = 1, 2, \dots$ ) in terms of Jackson integrals. The inductive structure of  $\varphi_n(x; s|q, t)$  can be reformulated as recurrence by Ruijsenaars-Macdonald operators of *row type*. By using this fact, we give an alternative proof of duality and bispectrality of  $f_n(x; s) = e_n(x; s)\varphi_n(x; s|q, t)$  which does not rely on the theory of Macdonald polynomials. We also present an explicit formula for  $\varphi_n(x; s|q, t)$  which manifestly shows (a part of) symmetry between  $x$  and  $s$  variables. We first recall from [8] the definition and some basic results on the family of Ruijsenaars-Macdonald operators of row type.

**5.1. Ruijsenaars-Macdonald operators of row type.** For each  $l = 0, 1, 2, \dots$ , we introduce the following  $q$ -difference operators  $H_l^x$  of *row type*:

$$H_l^x = \sum_{\nu \in \mathbb{N}^n; |\nu|=l} \prod_{1 \leq i < j \leq n} \frac{q^{\nu_i}x_i - q^{\nu_j}x_j}{x_i - x_j} \prod_{1 \leq i, j \leq n} \frac{(tx_i/x_j; q)_{\nu_i}}{(qx_i/x_j; q)_{\nu_i}} \prod_{i=1}^n T_{q, x}^{\nu_i}. \quad (5.1)$$

It is known by [8] that these operators satisfy the *Wronski relations*

$$\sum_{i+j=k} (-1)^i (1 - t^i q^j) D_i^x H_j^x = 0 \quad (k = 1, 2, \dots). \quad (5.2)$$

From this fact, it turns out that each  $H_l^x$  belong to the commutative ring  $\mathbb{C}[D_1^x, \dots, D_n^x]$  of Ruijsenaars-Macdonald operators. Hence we see that the operators  $H_l^x$  commute with each other, and that they commute with the operators  $D_r^x$  ( $r = 1, \dots, n$ ): For all  $k, l = 0, 1, 2, \dots$ ,

$$H_k^x H_l^x = H_l^x H_k^x, \quad H_k^x D_l^x = D_l^x H_k^x. \quad (5.3)$$

In terms of the two generating functions

$$D^x(u) = \sum_{r=0}^n (-u)^r D_r^x, \quad \text{and} \quad H^x(u) = \sum_{l=0}^{\infty} u^l H_l^x \quad (5.4)$$

for Ruijsenaars-Macdonald operators of column type and of row type, the Wronski relations above can be written as

$$D^x(u) H^x(u) = D^x(tu) H^x(qu). \quad (5.5)$$

From this relation, it follows that if  $f(x; s)$  satisfies the eigenfunction equation

$$D^x(u) f(x; s) = f(x; s) \prod_{i=1}^n (1 - us_i), \quad (5.6)$$

then it satisfies

$$H^x(u) f(x; s) = f(x; s) \prod_{i=1}^n \frac{(tus_i; q)_\infty}{(us_i; q)_\infty}. \quad (5.7)$$

We remark that the operator  $H^x(u)$  can be rewritten in the form

$$\begin{aligned} H^x(u) = & \prod_{1 \leq i < j \leq n} \frac{(qx_i/x_j; q)_\infty}{(qx_i/tx_j; q)_\infty} \sum_{\nu \in \mathbb{N}} u^{|\nu|} t^{\sum_i (i-1)\nu_i} \prod_{i=1}^n \frac{(t; q)_{\nu_i}}{(q; q)_{\nu_i}} \\ & \cdot \prod_{1 \leq i < j \leq n} \frac{(tx_i/x_j; q)_{\nu_i}}{(qx_i/x_j; q)_{\nu_i}} \frac{(q^{-\nu_j+1}x_i/tx_j; q)_{\nu_i}}{(q^{-\nu_j}x_i/x_j; q)_{\nu_i}} T_{q,x}^\nu \prod_{1 \leq i < j \leq n} \frac{(qx_i/tx_j; q)_\infty}{(qx_i/x_j; q)_\infty}. \end{aligned} \quad (5.8)$$

For each  $q$ -difference operator

$$A^x = A(x; T_{q,x}) = \sum_{\nu \in \mathbb{N}; |\nu| \leq m} A_\nu(x) T_{q,x}^\nu \in \mathbb{C}(x)[T_{q,x}^{\pm 1}], \quad (5.9)$$

we consider the  $q$ -difference operator

$$A^{x^{-1}} = A(x^{-1}, T_{q,x}^{-1}) = \sum_{\nu \in \mathbb{N}; |\nu| \leq m} A_\nu(x^{-1}) T_{q,x}^{-\nu} \quad (5.10)$$

obtained from  $A^x$  by inverting the variables  $x_i$  ( $i = 1, \dots, n$ ). As for Ruijsenaars-Macdonald  $q$ -difference operators of column type, one has

$$D_r^{x^{-1}} = t^{(n-1)r} D_{n-r}^x (D_n^x)^{-1} \quad (r = 0, 1, \dots, n). \quad (5.11)$$

Hence we see the commutative ring  $\mathbb{C}[D_1^x, \dots, D_{n-1}^x, (D_n^x)^{\pm 1}]$  of Ruijsenaars-Macdonald operators is stable by the operation  $A^x \rightarrow A^{x^{-1}}$ . By using this fact, one can show that the eigenfunction equation (5.6) implies

$$D^{x^{-1}}(u) f(x; s) = f(x; s) \prod_{i=1}^n (1 - t^{n-1}u/s_i), \quad (5.12)$$

and hence

$$H^{x^{-1}}(u) f(x; s) = f(x; s) \prod_{i=1}^n \frac{(t^n u/s_i; q)_\infty}{(t^{n-1}u/s_i; q)_\infty}, \quad (5.13)$$

namely,

$$H^{x^{-1}}(u/t^{n-1}) f(x; s) = f(x; s) \prod_{i=1}^n \frac{(tu/s_i; q)_\infty}{(u/s_i; q)_\infty}. \quad (5.14)$$

As we have seen in Section 1, it is convenient to transform the eigenfunction equation for  $f(x; s)$  into the equation for  $\psi(x; s)$  defined by

$$f(x; s) = x^\lambda \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \psi(x; s) \quad (5.15)$$

with parameter  $\lambda$  such that  $s = t^\delta q^\lambda$ . Then the equation (5.14) for  $f(x; s)$  is rewritten as

$$K^{(x;s)}(u) \psi(x; s) = \psi(x; s) \prod_{i=1}^n \frac{(tu/s_i; q)_\infty}{(u/s_i; q)_\infty}, \quad (5.16)$$

where

$$K^{(x;s)}(u) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} x^{-\lambda} H^{x^{-1}}(u/t^{n-1}) x^\lambda \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty}. \quad (5.17)$$

Form (5.8), this operator  $K^{(x;s)}(u)$  is determined as follows:

$$K^{(x;s)}(u) = \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n (u/s_i)^{\nu_i} \prod_{i=1}^n \frac{(t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \frac{(q^{-\nu_j+1}x_j/tx_i; q)_{\nu_i}}{(q^{-\nu_j}x_j/x_i; q)_{\nu_i}} T_{q,x}^{-\nu}. \quad (5.18)$$

We also use the notation  $K^{(x;s|q,t)}$  for this operator when we need to specify the parameter  $t$ .

**5.2. Recurrence by  $q$ -difference operators.** Let us consider the sequence of formal power series

$$\psi_n(s; x|q, t) = \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty} p_n(s; x|q, t) \quad (n = 1, 2, \dots) \quad (5.19)$$

so that

$$\varphi_n(s; x) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} \psi_n(s; x|q, t). \quad (5.20)$$

In Section 3, by the inductive construction by Jackson integrals, we proved  $x^\lambda \varphi_n(s; x|q, t)$  with  $s = t^\delta q^\lambda$  solves the eigenfunction equation (5.6) in  $x$  variables for  $n = 1, 2, \dots$ . Note that  $\psi_n(s; x|q, t)$  has leading coefficient 1 both in  $x$  variables and  $s$  variables, and hence we already know that  $\psi_n(s; x|q, t) = \psi_n(s; x|q, q/t)$  for all  $n = 1, 2, \dots$ .

An important observation is that the recurrence relation for  $\psi_n(s; x|q, t)$

$$\begin{aligned} \psi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(qs_{n+1}/ts_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \sum_{\nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(t; q)_{\nu_i}}{(q; q)_{\nu_i}} \prod_{1 \leq i < j \leq n+1} \frac{(tx_j/x_i; q)_{\nu_i}}{(qx_j/x_i; q)_{\nu_i}} \\ &\quad \cdot \prod_{1 \leq i < j \leq n} \frac{(q^{-\nu_j+1}x_j/tx_i; q)_{\nu_i}}{(q^{-\nu_j}x_j/x_i; q)_{\nu_i}} \prod_{i=1}^n (qs_{n+1}/ts_i)^{\nu_i} \psi_n(s; q^{-\nu}x|q, t) \end{aligned} \quad (5.21)$$

can be described in terms of the  $q$ -difference operator  $K^{(x;s|q,t)}(u)$  defined by (5.18) above. Note that this recurrence formula follows directly from that of  $p_n(x; s|q, t)$  in (3.11) by the definition (5.19). Also, we remark that this formula with  $t$  and  $q/t$  exchanged arose naturally in the previous section from the Jackson integral representation. In fact, the recurrence relation (5.21) can be expressed as

$$\begin{aligned} \psi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(tx_{n+1}/x_i; q)_\infty}{(qx_{n+1}/x_i; q)_\infty} \frac{(qs_{n+1}/ts_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \\ &\quad \cdot K^{(x;s|q,t)}(qs_{n+1}/t) \prod_{i=1}^n \frac{(qx_{n+1}/x_i; q)_\infty}{(tx_{n+1}/x_i; q)_\infty} \psi_n(s; x|q, t). \end{aligned} \quad (5.22)$$

By using this expression, we can show the duality relation  $\psi_n(x; s|q, t) = \psi_n(s; x|q, t)$  for  $\psi_n(s; x|q, t)$  ( $n = 1, 2, \dots$ ) by the induction on  $n$  starting from  $\psi_1(s; x|q, t) = 1$ . We

suppose that  $\psi_n(x; s|q, t) = \psi_n(s; x|q, t)$  as the induction hypothesis. Then we know that  $e_n(x; s)\varphi_n(s; x)$  is a formal solution of the eigenfunction equation in  $s$  variables, and hence

$$K^{(s; x|q, t)}(u) \psi(s; x|q, t) = \psi(s; x|q, t) \prod_{i=1}^n \frac{(tu/x_i; q)_\infty}{(u/x_i; q)_\infty}, \quad (5.23)$$

as well as

$$K^{(s; x|q, q/t)}(u) \psi(s; x|q, t) = \psi(s; x|q, t) \prod_{i=1}^n \frac{(qu/tx_i; q)_\infty}{(u/x_i; q)_\infty} \quad (5.24)$$

by the symmetry  $\psi(s; x|q, t) = \psi(s; x|q, q/t)$ . By formula (5.24) with  $u = tx_{n+1}$ , the recurrence formula (5.22) is rewritten in the form

$$\begin{aligned} \psi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(tx_{n+1}/x_i; q)_\infty}{(qx_{n+1}/x_i; q)_\infty} \frac{(qs_{n+1}/ts_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \\ &\quad \cdot K^{(x; s|q, t)}(qs_{n+1}/t) K^{(s; x|q, q/t)}(tx_{n+1}) \psi_n(s; x|q, t). \end{aligned} \quad (5.25)$$

By the symmetry  $\psi_n(s; x|q, t) = \psi_n(x; s|q, t) = \psi_n(x; s|q, q/t)$ , the right-hand side is symmetric with respect to exchanging  $x \leftrightarrow s$  and  $t \leftrightarrow q/t$  simultaneously. Hence we have

$$\psi_{n+1}(s; x|q, t) = \psi_{n+1}(x; s|q, q/t) = \psi_{n+1}(x; s|q, t), \quad (5.26)$$

as desired, by using the symmetry of  $\psi_{n+1}(s; x|q, t)$  with respect to  $t \leftrightarrow q/t$ .

The argument above, together with the recurrence by Jackson integrals in the previous section, provides a proof of duality and bispectrality of the formal power series

$$f_n(x; s|q, t) = e_n(x; s)\varphi_n(x; s|q, t) \in e_n(x; s)\mathbb{C}[[x^{-Q_+}]][[s^{-Q_+}]] \quad (5.27)$$

which does not depend on the theory of Macdonald polynomials.

**Theorem 5.1.** *The sequence of formal power series  $\psi_n(s; x|q, t)$  ( $n = 1, 2, \dots$ ) satisfies the following recurrence relation by the  $q$ -difference operators:*

$$\begin{aligned} \psi_{n+1}(s; x|q, t) &= \prod_{i=1}^n \frac{(tx_{n+1}/x_i; q)_\infty}{(qx_{n+1}/x_i; q)_\infty} \frac{(qs_{n+1}/ts_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \\ &\quad \cdot K^{(x; s|q, t)}(qs_{n+1}/t) K^{(s; x|q, q/t)}(tx_{n+1}) \psi_n(s; x|q, t), \end{aligned} \quad (5.28)$$

where  $K^{(x; s|q, t)}(u)$  is the  $q$ -difference operator defined by (5.18).

□

**Theorem 5.2.** *The formal power series  $\varphi_n(x; s|q, t)$  and  $\psi_n(x; s|q, t)$  satisfy the duality relation*

$$\varphi_n(x; s|q, t) = \varphi_n(s; x|q, t), \quad \psi_n(x; s|q, t) = \psi_n(s; x|q, t), \quad (5.29)$$

for  $n = 1, 2, \dots$ . Hence

$$\begin{aligned} f_n(x; s|q, t) &= e_n(x; s) \varphi_n(x; s; q|q, t) \\ &= e_n(x; s) \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} \psi_n(x; s|q, t) \end{aligned} \quad (5.30)$$

is a formal solution of the bispectral problem (4.1) for Ruijsenaars-Macdonald operators. Furthermore,  $\psi_n(x; s|q, t)$  is invariant under the change of parameters  $t \leftrightarrow q/t$ .

□

In explicit terms, the recurrence relation (5.25) means

$$\begin{aligned}
& \psi_{n+1}(s; x|q, t) \\
&= \prod_{i=1}^n \frac{(tx_{n+1}/x_i; q)_\infty}{(qx_{n+1}/x_i; q)_\infty} \frac{(qs_{n+1}/ts_i; q)_\infty}{(qs_{n+1}/s_i; q)_\infty} \\
& \cdot \sum_{\mu, \nu \in \mathbb{N}^n} \prod_{i=1}^n \frac{(t; q)_{\mu_i}}{(q; q)_{\mu_i}} \frac{(q/t; q)_{\nu_i}}{(q; q)_{\nu_i}} \\
& \cdot \prod_{1 \leq i < j \leq n} \frac{(tx_j/x_i; q)_{\mu_i}}{(qx_j/x_i; q)_{\mu_i}} \frac{(q^{-\mu_j} qx_j/tx_i; q)_{\mu_i}}{(q^{-\mu_j} x_j/x_i; q)_{\mu_i}} \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_{\nu_i}}{(qs_j/s_i; q)_{\nu_i}} \frac{(q^{-\nu_j} ts_j/s_i; q)_{\nu_i}}{(q^{-\nu_j} s_j/s_i; q)_{\nu_i}} \\
& \cdot \prod_{i=1}^n (qs_{n+1}/ts_i)^{\mu_i} (tx_{n+1}/x_i)^{\nu_i} q^{\sum_{i=1}^n \mu_i \nu_i} \psi_n(q^{-\nu} s; q^{-\mu} x|q, t). \tag{5.31}
\end{aligned}$$

This recurrence formula manifestly shows the symmetry of  $\psi_n(s; x|q, t)$  with respect to exchanging  $x \leftrightarrow s$  and  $t \leftrightarrow q/t$  simultaneously.

## 6. CONVERGENCE OF FORMAL SOLUTIONS

**6.1. Summary on formal solutions of the bispectral problem.** In previous sections, we investigated the joint bispectral problem

$$D^x(u)f(x; s) = f(x; s) \prod_{i=1}^n (1 - us_i), \quad D^s(u)f(x; s) = f(x; s) \prod_{i=1}^n (1 - ux_i), \tag{6.1}$$

in variables  $x = (x_1, \dots, x_n)$  and  $s = (s_1, \dots, s_n)$ , and constructed an explicit formal solution

$$\begin{aligned}
f_n(x; s|q, t) &= e_n(x; s) \varphi_n(x; s|q, t), \\
\varphi_n(x; s) &= \sum_{\mu, \nu \in Q_+} x^{-\mu} s^{-\nu} \varphi_{\mu, \nu} \in \mathbb{C}[[x^{-Q_+}]][[s^{-Q_+}]] \tag{6.2}
\end{aligned}$$

of this bispectral problem. Here we assume that  $e_n(x; s)$  is (possibly multi-valued) a meromorphic function on  $\mathbb{T}_x^n \times \mathbb{T}_s^n$ , satisfying the symmetry condition  $e_n(x; s) = e_n(s; x)$  and the  $q$ -difference equations

$$T_{q, x_i}(e_n(x; s)) = e_n(x; s) s_i / t^{n-i}, \quad T_{q, s_i}(e_n(x; s)) = e_n(x; s) x_i / t^{n-i} \quad (i = 1, \dots, n). \tag{6.3}$$

We introduced another formal power series

$$\psi_n(x; s|q, t) = \sum_{\mu, \nu \in Q_+} x^{-\mu} s^{-\nu} \psi_{\mu, \nu} \in \mathbb{C}[[x^{-Q_+}]][[s^{-Q_+}]] \tag{6.4}$$

such that

$$\varphi_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(qx_j/tx_i; q)_\infty} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} \psi_n(x; s|q, t). \tag{6.5}$$

This formal power series  $\psi_n(x; s|q, t)$  satisfies the initial conditions

$$\psi_n(x; s|q, t)|_{x_{i+1}/x_i=0 \ (i=1, \dots, n-1)} = 1, \quad \psi_n(x; s|q, t)|_{s_{i+1}/s_i=0 \ (i=1, \dots, n-1)} = 1, \tag{6.6}$$

and remarkable symmetry relations

$$\psi_n(x; s|q, t) = \psi_n(s; x|q, t), \quad \psi_n(x; s|q, t) = \psi_n(x; s|q, q/t). \quad (6.7)$$

The basic object in our framework was the formal power series

$$p_n(x; s|q, t) = \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{ij}} \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]], \quad (6.8)$$

where

$$\begin{aligned} c_n(\theta; s|q, t) &= \prod_{k=2}^n \prod_{1 \leq i < j \leq k} \frac{(q^{\sum_{a>k}(\theta_{i,a}-\theta_{j,a})} t s_j / s_i; q)_{\theta_{i,k}}}{(q^{\sum_{a>k}(\theta_{i,a}-\theta_{j,a})} q s_j / s_i; q)_{\theta_{i,k}}} \\ &\cdot \prod_{k=2}^n \prod_{1 \leq i \leq j < k} \frac{(q^{-\theta_{j,k}+\sum_{a>k}(\theta_{i,a}-\theta_{j,a})} q s_j / t s_i; q)_{\theta_{i,k}}}{(q^{-\theta_{j,k}+\sum_{a>k}(\theta_{i,a}-\theta_{j,a})} s_j / s_i; q)_{\theta_{i,k}}} \quad (\theta \in M_n). \end{aligned} \quad (6.9)$$

By means of this  $p_n(x; s|q, t)$ ,  $\varphi_n(x; s|q, t)$  and  $\psi_n(x; s|q, t)$  are expressed as

$$\varphi_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(q s_j / s_i; q)_{\infty}}{(q s_j / t s_i; q)_{\infty}} p_n(x; s|q, t) \quad (6.10)$$

and

$$\psi_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(q x_j / t x_i; q)_{\infty}}{(q x_j / x_i; q)_{\infty}} p_n(x; s|q, t). \quad (6.11)$$

By the symmetry (6.7), this formal power series  $\psi_n(x; s|q, t)$  can be expressed in four ways:

$$\begin{aligned} \psi_n(x; s|q, t) &= \prod_{1 \leq i < j \leq n} \frac{(q x_j / t x_i; q)_{\infty}}{(q x_j / x_i; q)_{\infty}} p_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(q s_j / t s_i; q)_{\infty}}{(q s_j / s_i; q)_{\infty}} p_n(s; x|q, t) \\ &= \prod_{1 \leq i < j \leq n} \frac{(t x_j / x_i; q)_{\infty}}{(q x_j / x_i; q)_{\infty}} p_n(x; s|q, q/t) = \prod_{1 \leq i < j \leq n} \frac{(t s_j / s_i; q)_{\infty}}{(q s_j / s_i; q)_{\infty}} p_n(s; x|q, q/t). \end{aligned} \quad (6.12)$$

**6.2. Convergence of formal solutions.** In view of symmetry, we will mainly use below the power series  $\psi_n(x; s|q, t)$  so that

$$\begin{aligned} f_n(x; s|q, t) &= e_n(x; s) \varphi_n(x; s|q, t) \\ &= e_n(x; s) \prod_{1 \leq i < j \leq n} \frac{(q x_j / x_i; q)_{\infty}}{(q x_j / t x_i; q)_{\infty}} \frac{(q s_j / s_i; q)_{\infty}}{(q s_j / t s_i; q)_{\infty}} \psi_n(x; s|q, t). \end{aligned} \quad (6.13)$$

From the initial conditions (6.6), we already know

$$\psi_n(x; s|q, t) \in \mathbb{C}(s^{-Q_+})[[x^{-Q_+}]] \cap \mathbb{C}(x^{-Q_+})[[s^{-Q_+}]]. \quad (6.14)$$

As to the expansion

$$\psi_n(x; s|q, t) = \sum_{\mu \in Q_+} x^{-\mu} \psi_{\mu}(s), \quad \psi_{\mu}(s) \in \mathbb{C}(s^{-Q_+}), \quad (6.15)$$

the  $\psi_{\mu}(s)$  are rational functions in  $(s_2/s_1, \dots, s_n/s_{n-1})$ , regular at  $(s_2/s_1, \dots, s_n/s_{n-1}) = 0$ , and have at most simple poles along

$$s_j/s_i = q^{-k-1} \quad (1 \leq i < j \leq n; \quad k = 0, 1, 2, \dots). \quad (6.16)$$



On the other hand,  $\psi_n(x; s|q, t)$  is expressed as

$$\begin{aligned}\psi(x; s|q, t) &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} p_n(x; s|q, t) \\ &= \prod_{1 \leq i < j \leq n} \frac{(qx_j/tx_i; q)_\infty}{(qx_j/x_i; q)_\infty} \left( \sum_{\mu \in Q_+} x^{-\mu} p_\mu(s) \right),\end{aligned}\tag{6.17}$$

where

$$p_\mu(s) = \sum_{\theta \in M_n(\mu)} c_n(\theta; s|q, t).\tag{6.18}$$

Note that these  $c_n(\theta; s|q, t)$  may have multiple poles along

$$s_j/s_i = q^k \quad (1 \leq i < j \leq n; k \in \mathbb{Z}).\tag{6.19}$$

We denote by  $\mathbb{C}_z^{n-1}$  the  $(n-1)$ -dimensional affine space with canonical coordinates  $z = (z_1, \dots, z_{n-1})$ , and define a holomorphic mapping  $\pi : \mathbb{T}_x^n \rightarrow \mathbb{C}_z^{n-1}$  by

$$\pi(a) = (a_2/a_1, \dots, a_n/a_{n-1}) \quad \text{for each } a = (a_1, \dots, a_n) \in \mathbb{T}_x^n,\tag{6.20}$$

so that  $\pi^*(z_i) = x_{i+1}/x_i$  ( $i = 1, \dots, n$ ), where  $\pi^* : \mathcal{O}_{\mathbb{C}^{n-1}} \rightarrow \pi_*(\mathcal{O}_{\mathbb{T}^n})$  denotes the pull-back by  $\pi$  in the sense of sheaves of holomorphic functions. Similarly, for the  $s$  variables, we use the  $(n-1)$ -dimensional affine space  $\mathbb{C}_w^{n-1}$  with canonical coordinates  $w = (w_1, \dots, w_{n-1})$  such that  $\pi^*(w_i) = s_{i+1}/s_i$  ( $i = 1, \dots, n-1$ ).

In view of the singularity of  $p_\mu(s)$ , we define an open subset  $D_w \subset \mathbb{C}_w^{n-1}$  by

$$D_w = \{w = (w_1, \dots, w_{n-1}) \in \mathbb{C}_w^{n-1} \mid w_i \cdots w_{j-1} \notin q^{-\mathbb{Z}} \cup \{0\} \quad (1 \leq i < j \leq n)\},\tag{6.21}$$

so that

$$\pi^{-1}(D_w) = \{s = (s_1, \dots, s_n) \in \mathbb{T}_s^n \mid s_j/s_i \notin q^{-\mathbb{Z}} \quad (1 \leq i < j \leq n)\}.\tag{6.22}$$

For each  $r > 0$  we set

$$\begin{aligned}U_z(r) &= \{z = (z_1, \dots, z_{n-1}) \in \mathbb{C}_z^{n-1} \mid |z_i| < r \quad (i = 1, \dots, n-1)\}, \\ B_z(r) &= \{z = (z_1, \dots, z_{n-1}) \in \mathbb{C}_z^{n-1} \mid |z_i| \leq r \quad (i = 1, \dots, n-1)\},\end{aligned}\tag{6.23}$$

so that

$$\pi^{-1}(B_z(r)) = \{x = (x_1, \dots, x_n) \in \mathbb{T}_x^n \mid |x_j/x_i| \leq r^{j-i} \quad (1 \leq i < j \leq n)\}.\tag{6.24}$$

**Proposition 6.1.** *For  $n = 2, 3, \dots$ , we regard  $p_n(x; s|q, t)$  as a formal power series in  $z = (z_1, \dots, z_{n-1})$  with coefficients in  $\mathcal{O}(D_w)$ :*

$$p_n(x; s|q, t) = \sum_{\theta \in M_n} c_n(\theta; s|q, t) \prod_{1 \leq i < j \leq n} (x_j/x_i)^{\theta_{i,j}} \in \mathcal{O}(D_w)[[z]]\tag{6.25}$$

We set  $r_0 = |q/t|^{\frac{n-2}{n-1}}$  if  $|q/t| \leq 1$ , and  $r_0 = |t/q|$  if  $|q/t| \geq 1$ . Then for any compact subset  $K \subset D_w$  and for any  $r < r_0$ , this series (6.25) is absolutely convergent, uniformly on  $B_z(r) \times K$ . Hence  $p_n(x; s|q, t)$  defines a holomorphic function on  $U_z(r_0) \times D_w$ .

Note that  $c_n(\theta; s|q, t)$  is a product of factors of the form

$$\frac{(q^l au; q)_k}{(q^l u; q)_k} \quad (k \in \mathbb{N}, l \in \mathbb{Z}), \quad (6.26)$$

with  $(u, a) = (qs_j/s_i, t/q)$  or  $(u, a) = (s_j/s_i, q/t)$ .

**Lemma 6.2.** *Let  $a \in \mathbb{C}^*$  be a nonzero constant. For any compact subset  $L \subset \mathbb{C}^* \setminus q^{\mathbb{Z}}$ , there exists a positive constant  $C_L > 0$  such that, for any finite subset  $I \subset \mathbb{Z}$ ,*

$$\left| \prod_{i \in I} \frac{1 - q^i au}{1 - q^i u} \right| \leq C_L \max\{|a|, 1\}^{|I|} \quad (u \in L). \quad (6.27)$$

In particular,

$$\left| \frac{(q^l au; q)_k}{(q^l u; q)_k} \right| \leq C_L \max\{|a|, 1\}^k \quad (u \in L) \quad (6.28)$$

for any  $k \in \mathbb{N}$  and  $l \in \mathbb{Z}$ .

*Proof.* We take nonnegative integers  $M, N \in \mathbb{N}$  such that  $|q|^{M+1} < |u| < |q|^{-N}$  for all  $u \in L$ . Given a finite subset  $I \subset \mathbb{Z}$ , we divide  $I$  into three parts  $I = I_- \sqcup I_0 \sqcup I_+$  by setting

$$I_- = I \cap (-\infty, -M-1], \quad I_0 = I \cap [-M, N-1], \quad I_+ = I \cap [N, +\infty). \quad (6.29)$$

Suppose  $u \in L$ . Since  $|q^i u| < 1$  for all  $i \geq N$ , we have

$$\left| \prod_{i \in I_+} \frac{1 - q^i au}{1 - q^i u} \right| \leq \prod_{i \in I_+} \frac{1 + |q^i au|}{1 - |q^i u|} \leq \frac{(-|q^N au|; |q|)_{\infty}}{(|q^N u|; |q|)_{\infty}}. \quad (6.30)$$

Since  $|q^{-i}/u| < 1$  for  $i \leq -M-1$ ,

$$\left| \prod_{i \in I_-} \frac{1 - q^i au}{1 - q^i u} \right| = |a|^{|I_-|} \prod_{i \in I_-} \frac{1 + |q^{-i}/au|}{1 - |q^{-i}/u|} \leq |a|^{|I_-|} \frac{(-|q^{M+1}/au|; |q|)_{\infty}}{(|q^{M+1}/u|; |q|)_{\infty}}. \quad (6.31)$$

Hence by using three continuous functions

$$c_+(u) = \frac{(-|q^N au|; |q|)_{\infty}}{(|q^N u|; |q|)_{\infty}}, \quad c_-(u) = \frac{(-|q^{M+1}/au|; |q|)_{\infty}}{(|q^{M+1}/u|; |q|)_{\infty}}, \quad (6.32)$$

and

$$c_0(u) = \max_{J \subseteq [-M, N-1]} \left| \prod_{i \in J} \frac{1 - q^i au}{1 - q^i u} \right|, \quad (6.33)$$

we estimate

$$\left| \prod_{i \in I} \frac{1 - q^i au}{1 - q^i u} \right| \leq |a|^{|I_-|} c_-(u) c_0(u) c_+(u). \quad (6.34)$$

Since  $|a|^{|I_-|} \leq |a|^{|I|}$  if  $|a| \geq 1$ , we obtain the estimate of Lemma by taking for  $C_L$  the maximum of the continuous function  $c_-(u)c_0(u)c_+(u)$  on  $L$ .  $\square$

*Proof of Proposition 6.1.* Let  $K$  be any compact subset of  $D_w$ . By the definition (3.3),

$$\begin{aligned}
c_n(\theta; s|q, t) &= \prod_{1 \leq i < k \leq n} (q/t)^{\theta_{i,k}} \frac{(t; q)_{\theta_{i,k}}}{(q; q)_{\theta_{i,k}}} \\
&\cdot \prod_{1 \leq i < j \leq k \leq n} \frac{(q^{\sum_{a>k}(\theta_{i,a}-\theta_{j,a})} t s_j / s_i; q)_{\theta_{i,k}}}{(q^{\sum_{a>k}(\theta_{i,a}-\theta_{j,a})} q s_j / s_i; q)_{\theta_{i,k}}} \\
&\cdot \prod_{1 \leq i < j < k \leq n} \frac{(q^{-\theta_{j,k} + \sum_{a>k}(\theta_{i,a}-\theta_{j,a})} q s_j / t s_i; q)_{\theta_{i,k}}}{(q^{-\theta_{j,k} + \sum_{a>k}(\theta_{i,a}-\theta_{j,a})} s_j / s_i; q)_{\theta_{i,k}}} \quad (\theta \in M_n). \tag{6.35}
\end{aligned}$$

Suppose that  $|q/t| \leq 1$ . Then by Lemma 6.2 we see there exists a positive constant  $C_K$  such that

$$|c_n(\theta; s|q, t)| \leq C_K \prod_{1 \leq i < j \leq n} \frac{(-|t|; |q|)_{\theta_{i,j}}}{(|q|; |q|)_{\theta_{i,j}}} \prod_{1 \leq i < j \leq n} |t/q|^{(j-i-1)\theta_{i,j}} \quad (s \in \pi^{-1}(K)). \tag{6.36}$$

Hence

$$\begin{aligned}
\sum_{\theta \in M_n} |c_n(\theta; q, t)| \prod_{1 \leq i < j \leq n} |x_j/x_i|^{\theta_{ij}} &\leq C_K \sum_{\theta \in M_n} \prod_{1 \leq i < j \leq n} \frac{(-|t|; |q|)_{\theta_{i,j}}}{(|q|; |q|)_{\theta_{i,j}}} |(t/q)^{j-i-1} x_j/x_i|^{\theta_{ij}} \\
&= C_K \prod_{1 \leq i < j \leq n} \frac{(-|(t/q)^{j-i-1} t x_j/x_i|; |q|)_{\infty}}{(|(t/q)^{j-i-1} x_j/x_i|; |q|)_{\infty}} \tag{6.37}
\end{aligned}$$

if  $|x_j/x_i| < |q/t|^{j-i-1}$  for all  $1 \leq i < j \leq n$ . Hence for any  $r > 0$  such that  $r^k < |q/t|^{k-1}$  for  $k = 1, \dots, n-1$ , the function series  $p_n(x; s|q, t)$  is absolutely convergent, uniformly on  $B_z(r) \times K$ . The condition above for  $r$  is equivalent to  $r < r_0$ ,  $r_0 = |q/t|^{\frac{n-2}{n-1}}$ . The case  $|q/t| \geq 1$  can be treated in a similar way.  $\square$

By Proposition 6.1, the function

$$\psi(x; s|q, t) = \prod_{1 \leq i < j \leq n} \frac{(q x_j / t x_i; q)_{\infty}}{(q x_j / x_i; q)_{\infty}} p_n(x; s|q, t) \tag{6.38}$$

is holomorphic on  $U_z(r_0) \times D_w$ . Hence, the Taylor expansion of  $\psi_n(x; s|q, t)$

$$\psi_n(x; s|q, t) = \sum_{\mu \in Q_+} x^{-\mu} \psi_{\mu}(s) \in \mathcal{O}(D_w)[[z]] \tag{6.39}$$

in  $z$  variables is normally convergent in  $U_z(r_0) \times D_w$ , namely, absolutely convergent, uniformly on any compact subset of  $U_z(r_0) \times D_w$ .

We already know that the expansion coefficients  $\psi_{\mu}(s)$  ( $\mu \in Q_+$ ) are rational functions in  $w = (s_2/s_1, \dots, s_n/s_{n-1})$ , regular at  $w = 0$ , and have at most simple poles along  $s_j/s_i = w_i \cdots w_{j-1} = q^{-k-1}$  ( $1 \leq i < j \leq n$ ;  $k = 0, 1, 2, \dots$ ). In order to eliminate these poles of the coefficients, we introduce the function

$$F_n(x; s|q, t) = \prod_{1 \leq i < j \leq n} (q x_j / x_i; q)_{\infty} (q s_j / s_i; q)_{\infty} \psi_n(x; s|q, t). \tag{6.40}$$

As the formal power series in  $z$  variables, the expansion coefficients are entire holomorphic functions in  $s$  variables:

$$F_n(x; s|q, t) = \sum_{\mu \in Q_+} x^{-\mu} F_\mu(s) \in \mathcal{O}(\mathbb{C}_w^{n-1})[[z]]. \quad (6.41)$$

On the other hand, this function  $F_n(x; s|q, t)$  is holomorphic on  $U_z(r_0) \times D_w$ , and hence, for any compact subset  $K \subset D_w$ , away from the divisors  $s_j/s_i \in q^{\mathbb{Z}}$  ( $1 \leq i < j \leq n$ ), and for any  $r < r_0$ , this Taylor expansion is absolutely convergent, uniformly on  $B_z(r) \times K$ . Since the maximum of  $|x^{-\mu}|$  on  $B_z(r)$  is given by  $r^{(\delta, \mu)}$ , it means that

$$\sum_{\mu \in Q_+} |x^{-\mu}| |F_\mu(s)| \leq \sum_{\mu \in Q_+} r^{(\delta, \mu)} \|F_\mu\|_K < +\infty \quad (6.42)$$

on  $B_z(r) \times K$ , where  $\|F_\mu\|_K$  stands for the supremum norm on the compact set  $K$  with  $F_\mu(s)$  regarded as a function in  $w$  variables.

**Lemma 6.3.** *For any compact subset  $K \subset \mathbb{C}_w^{n-1}$ , and for any  $r < r_0$ , the Taylor expansion (6.41) of  $F_n(x; s|q, t)$  in  $z$  variables is absolutely convergent, uniformly on  $B_z(r) \times K$ . Hence it defines a holomorphic function on  $U_z(r_0) \times \mathbb{C}_w^{n-1}$ .*

*Proof.* Let  $\rho > 0$  be any irrational number, and set

$$K_\rho = \{w = (w_1, \dots, w_{n-1}) \in \mathbb{C}_w^{n-1} \mid |w_i| = |q|^{-\rho} \ (i = 1, \dots, n-1)\} \quad (6.43)$$

Then we have  $K_\rho \subset D_w$ , since  $|s_j/s_i| = |z_i \cdots z_{j-1}| = |q|^{-(j-i)\rho} \notin |q|^{\mathbb{Z}}$  for any  $i, j$  with  $1 \leq i < j \leq n$ . Consider the corresponding closed polydisc in  $\mathbb{C}_w^{n-1}$ :

$$B_w(|q|^{-\rho}) = \{w = (w_1, \dots, w_{n-1}) \in \mathbb{C}_w^{n-1} \mid |w_i| \leq |q|^{-\rho} \ (i = 1, \dots, n-1)\}. \quad (6.44)$$

Then for each holomorphic function  $F_\mu(s) \in \mathcal{O}(C_w^{n-1})$  ( $\mu \in Q_+$ ), the maximum of its absolute values on  $B_w(|q|^{-\rho})$  is attained on the Silov boundary  $K_\rho$ , namely,  $\|F_\mu\|_{B_w(|q|^{-\rho})} = \|F_\mu\|_{K_\rho}$ . This implies that, for any  $r < r_0$ ,

$$\sum_{\mu \in Q_+} |x|^{-\mu} |F_\mu(s)| \leq \sum_{\mu \in Q_+} r^{(\delta, \mu)} \|F_\mu\|_{K_\rho} < +\infty, \quad (6.45)$$

uniformly on  $B_z(r) \times B_w(|q|^{-\rho})$ . Since  $\rho > 0$  can be taken arbitrarily large, the Taylor expansion (6.41) of  $F_n(x; s|q, t)$  in  $z$  variables is absolutely convergent, on any compact subset in  $U_z(r_0) \times \mathbb{C}_w^{n-1}$ .  $\square$

**Proposition 6.4.** *As to the function  $F_n(x; s|q, t)$  defined by (6.40), its Taylor series*

$$F_n(x; s|q, t) = \sum_{\mu, \nu \in Q_+} x^{-\mu} s^{-\nu} F_{\mu, \nu} \in \mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]] \quad (6.46)$$

*is normally convergent in  $\mathbb{C}_z^{n-1} \times \mathbb{C}_w^{n-1}$ . Hence,  $F_n(x; s|q, t)$  is continued to an entire holomorphic function on  $\mathbb{C}_z^{n-1} \times \mathbb{C}_w^{n-1}$ .*

*Proof.* We know that this Taylor series is normally convergent in  $U_z(r_0) \times \mathbb{C}_w^{n-1}$ . From the symmetry  $F_n(x; s|q, t) = F_n(s; x|q, t)$ , it is also normally convergent in  $\mathbb{C}_z^{n-1} \times U_w(r_0)$ . The domain of convergence of this power series contains both  $U_z(r_0) \times \mathbb{C}_w^{n-1}$  and  $\mathbb{C}_z^{n-1} \times U_w(r_0)$ , and hence it must be the whole affine space  $\mathbb{C}_z^{n-1} \times \mathbb{C}_w^{n-1}$  by the logarithmic convexity of the domain of convergence. In fact, consider the compact subset  $B_z(a) \times B_w(b)$  for

arbitrary  $a, b > 0$ . Take sufficiently large  $c > 0$  so that  $a/c < r_0$  and  $b/c < r_0$ . Then we have

$$a^{\langle \delta, \mu \rangle} b^{\langle \delta, \nu \rangle} \leq \frac{1}{2} \left( (a/c)^{\langle \delta, \mu \rangle} (bc)^{\langle \delta, \nu \rangle} + (ac)^{\langle \delta, \mu \rangle} (b/c)^{\langle \delta, \nu \rangle} \right) \quad (\mu, \nu \in Q_+), \quad (6.47)$$

and hence

$$\begin{aligned} \sum_{\mu, \nu \in Q_+} |x^{-\mu}| |s^{-\nu}| |F_{\mu, \nu}| &\leq \sum_{\mu, \nu \in Q_+} a^{\langle \delta, \mu \rangle} b^{\langle \delta, \nu \rangle} |F_{\mu, \nu}| \\ &\leq \frac{1}{2} \left( \sum_{\mu, \nu \in Q_+} (a/c)^{\langle \delta, \mu \rangle} (bc)^{\langle \delta, \nu \rangle} |F_{\mu, \nu}| + \sum_{\mu, \nu \in Q_+} (ac)^{\langle \delta, \mu \rangle} (b/c)^{\langle \delta, \nu \rangle} |F_{\mu, \nu}| \right) < +\infty, \end{aligned} \quad (6.48)$$

on  $B_z(a) \times B_w(b)$ .  $\square$

**Theorem 6.5.** *For the formal solution*

$$f_n(x; s|q, t) = e_n(x; s) \varphi_n(x; s|q, t), \quad \varphi_n(x; s|q, t) \in \mathbb{C}[[x^{-Q_+}]] [[s^{-Q_+}]] \quad (6.49)$$

of the bispectral problem (6.1) for Ruijsenaars-Macdonald operators, introduce a formal power series  $F_n(x; s|q, t)$  by setting

$$\varphi_n(x; s|q, t) = \frac{F_n(x; s|q, t)}{\prod_{1 \leq i < j \leq n} (qx_j/tx_i; q)_\infty (qs_j/ts_i; q)_\infty}. \quad (6.50)$$

Then,  $F_n(x; s|q, t)$  represents a holomorphic function on  $\mathbb{C}_z^{n-1} \times \mathbb{C}_w^{n-1}$  in the variable  $(z; w) = (z_1, \dots, z_{n-1}; w_1, \dots, w_{n-1})$  with  $z_i = x_{i+1}/x_i$ ,  $w_i = s_{i+1}/s_i$  ( $i = 1, \dots, n-1$ ), depending holomorphically on  $t \in \mathbb{C}^*$ . Furthermore, it satisfies the symmetry conditions

$$F_n(x; s|q, t) = F_n(s; x|q, t), \quad F_n(x; s|q, t) = F_n(x; s|q, q/t). \quad (6.51)$$

$\square$

For  $n = 2$ ,  $F_2(x; s|q, t)$  is expressed as

$$F_2(x; s|q, t) = (t; q)_\infty (qx_2s_2/x_1s_1; q)_\infty {}_2\phi_1 \left[ \begin{matrix} qx_2/tx_1, qs_2/ts_1 \\ qx_2s_2/x_1s_1 \end{matrix}; q, t \right] \quad (|t| < 1) \quad (6.52)$$

as we already remarked in Section 3, and by symmetry,

$$F_2(x; s|q, t) = (q/t; q)_\infty (qx_2s_2/x_1s_1; q)_\infty {}_2\phi_1 \left[ \begin{matrix} tx_2/x_1, ts_2/s_1 \\ qx_2s_2/x_1s_1 \end{matrix}; q, q/t \right] \quad (|q/t| < 1). \quad (6.53)$$

**6.3. Principal specialization.** We give a remark on the evaluation of  $\varphi_n(x; s|q, t)$  at  $x = t^\delta$ . By duality, this question is equivalent to knowing the value at  $s = t^\delta$ . The recurrence formula (3.11) for  $s = t^\delta$  reduces to a single term with  $\nu = 0$  because of the existence of the factor

$$\prod_{1 \leq i < j \leq n+1} \frac{(ts_j/s_i; q)_{\nu_i}}{(qs_j/s_i; q)_{\nu_i}}, \quad (6.54)$$

since  $(ts_j/s_i; q)_{\nu_i}$  for  $j = i + 1$  vanishes unless  $\nu_i = 0$ . Hence  $p_n(x; t^\delta|q, t) = 1$  for all  $n = 1, 2, \dots$ . From

$$\varphi_n(x; t^\delta|q, t) = \prod_{1 \leq i < j \leq n} \frac{(qt^{i-j}; q)_\infty}{(qt^{i-j-1}; q)_\infty} = \prod_{i=1}^n \frac{(q/t; q)_\infty}{(q/t^i; q)_\infty}. \quad (6.55)$$

we obtain

$$\varphi_n(t^\delta; s|q, t) = \varphi_n(x; t^\delta|q, t) = \prod_{i=1}^n \frac{(q/t; q)_\infty}{(q/t^i; q)_\infty}. \quad (6.56)$$

According to the definition (3.12), this implies a nontrivial summation formula:

**Theorem 6.6.** *Let  $|t| > |q|^{-(n-2)}$ . We have*

$$\sum_{\theta \in M_n} c_n(\theta; s|q, t) t^{\sum_{i < j} (i-j)\theta_{ij}} = \prod_{i=1}^n \frac{(q/t; q)_\infty}{(q/t^i; q)_\infty} \prod_{1 \leq i < j \leq n} \frac{(qs_j/ts_i; q)_\infty}{(qs_j/s_i; q)_\infty}. \quad (6.57)$$

*Proof.* By Proposition 6.1, when  $|t| > |q|^{-(n-2)}$ , the left-hand side is normally convergent for  $s_j/s_i \notin q^{\mathbb{Z}}$  ( $1 \leq i < j \leq n$ ).  $\square$

## 7. CASE $n = 3$

**7.1. Results.** In this section, we present a direct method to check the structure of the divisor of the poles of  $p_n(x; s|q, t)$  and the duality for the case  $n = 3$ , by applying several transformation and summation formulas for basic hypergeometric series. Note that the duality (6.7) can be stated as

$$\prod_{1 \leq i < j \leq n} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} p_n(x; s|q, q/t) = \prod_{1 \leq i < j \leq n} \frac{(qx_j/x_i; q)_\infty}{(tx_j/x_i; q)_\infty} p_n(s; x|q, t). \quad (7.1)$$

**Theorem 7.1.** *We have*

$$\begin{aligned} p_3(x; s|q, q/t) &= \sum_{\theta \in M^{(3)}} c_3(\theta; s_1, s_2, s_3|q, t) (x_2/x_1)^{\theta_{1,2}} (x_3/x_1)^{\theta_{1,3}} (x_3/x_2)^{\theta_{2,3}} \\ &= \sum_{k=0}^{\infty} \frac{(q/t; q)_k (q/t; q)_k (t; q)_k (t; q)_k}{(q; q)_k (qs_2/s_1; q)_k (qs_3/s_1; q)_k (qs_3/s_2; q)_k} (qs_3/ts_1)^k (tx_3/x_1)^k \\ &\quad \times \prod_{1 \leq i < j \leq 3} {}_2\phi_1 \left[ \begin{matrix} q^{k+1}/t, qs_j/ts_i \\ q^{k+1}s_j/s_i \end{matrix}; q, tx_j/x_i \right]. \end{aligned} \quad (7.2)$$

This manifestly shows that  $p_3(x; s|q, q/t)$  has at most simple poles along the divisors  $s_j/s_i = q^{-k-1}$  ( $1 \leq i < j \leq 3; k = 0, 1, 2, \dots$ ).

A proof will be given in Section 7.3. We can recast this into another form.

**Theorem 7.2.** *We have*

$$\begin{aligned} \varphi_3(x, s|q, t) &= \prod_{1 \leq i < j \leq 3} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} p_3(x; s|q, t) \\ &= \sum_{k \geq 0} \frac{(q/t; q)_k (q/t; q)_k}{(q; q)_k (t; q)_k} (qx_3s_3/x_1s_1)^k \\ &\quad \times \prod_{1 \leq i < j \leq 3} \frac{(t; q)_\infty (qx_js_j/x_is_i; q)_\infty}{(qx_j/tx_i; q)_\infty (qs_j/ts_i; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} qx_j/tx_i, qs_j/ts_i \\ qx_js_j/x_is_i \end{matrix}; q, q^k t \right] \quad (|t| < 1), \end{aligned} \quad (7.3)$$

which manifestly shows the duality  $\varphi_3(x, s|q, t) = \varphi_3(s, x|q, t)$ .

*Proof.* We can proceed in a completely parallel manner as (4.11) for  $n = 2$ , by using the  $q$ -binomial theorem and Theorem 7.1.  $\square$

There is yet another way to see the duality manifestly. From Theorem 7.1 and the  $q$ -binomial theorem, we have

$$\begin{aligned}
& \prod_{1 \leq i < j \leq 3} \frac{(qs_j/s_i; q)_\infty}{(qs_j/ts_i; q)_\infty} p_3(x; s|q, q/t) \\
&= \sum_{\substack{k \geq 0 \\ \theta \in M^{(3)}}} \frac{(q/t; q)_k (q/t; q)_k (t; q)_k (t; q)_k}{(q; q)_k} (qs_3/ts_1)^k (tx_3/x_1)^k \\
&\times \prod_{1 \leq i < j \leq 3} \frac{(q^{k+1}/t; q)_{\theta_{i,j}}}{(q; q)_{\theta_{i,j}}} \frac{(q^{k+\theta_{i,j}+1}s_j/s_i; q)_\infty}{(q^{\theta_{i,j}+1}s_j/ts_i; q)_\infty} (tx_j/x_i)^{\theta_{i,j}} \\
&= \sum_{\substack{k \geq 0 \\ \theta, \nu \in M^{(3)}}} \frac{(q/t; q)_k (q/t; q)_k (t; q)_k (t; q)_k}{(q; q)_k} (qs_3/ts_1)^k (tx_3/x_1)^k \\
&\times \prod_{1 \leq i < j \leq 3} \frac{(q^{k+1}/t; q)_{\theta_{i,j}}}{(q; q)_{\theta_{i,j}}} \frac{(q^k t; q)_{\nu_{i,j}}}{(q; q)_{\nu_{i,j}}} (tx_j/x_i)^{\theta_{i,j}} (qs_j/ts_i)^{\nu_{i,j}} q^{\theta_{i,j}\nu_{i,j}},
\end{aligned} \tag{7.4}$$

which indicates the duality of the form stated in (7.1).

**7.2. Some transformation formulas.** For our proof of Theorem 7.1, we prepare several transformation formulas. We use the notations used in [3].

**Proposition 7.3.** *Let  $\theta, r$  be nonnegative integers. We have the transformation formula for very well-poised series*

$$\begin{aligned}
& {}_{r+9}W_{r+8}(a; q^{-\theta}, q^\theta af, a_1, \dots, a_r, \left(\frac{aq}{f}\right)^{\frac{1}{2}}, -\left(\frac{aq}{f}\right)^{\frac{1}{2}}, \left(\frac{aq^2}{f}\right)^{\frac{1}{2}}, -\left(\frac{aq^2}{f}\right)^{\frac{1}{2}}; q, z) \\
&= \frac{(aq, f^2/q; q)_\theta}{(af, f; q)_\theta} \sum_{m \geq 0} \frac{(q/f, q^{-\theta}, aq/f; q)_m}{(q, q^{-\theta}q^2/f^2, aq; q)_m} q^m {}_{r+5}W_{r+4}(a; q^{-m}, q^m aq/f, a_1, \dots, a_r; q, z).
\end{aligned} \tag{7.5}$$

*Proof.* Note that from the  $q$ -Saalschütz summation formula [3], we have

$$\begin{aligned}
& (f/q)^k \frac{(af, f; q)_\theta}{(aq, f^2/q; q)_\theta} \frac{(q^{-\theta}q^2/f^2, q^\theta af; q)_k (aq; q)_{2k}}{(q^{-\theta}q/f, q^\theta aq; q)_k (af; q)_{2k}} \\
&= \frac{(f, q^{2k}af; q)_{\theta-k}}{(f^2/q, q^{2k}aq; q)_{\theta-k}} \\
&= {}_3\phi_2 \left[ \begin{matrix} q^{2k}aq/f, q/f, q^{-\theta+k} \\ q^{2k}aq, q^{-\theta+k}q^2/f^2 \end{matrix} ; q, q \right].
\end{aligned} \tag{7.6}$$

By using this, we can proceed as follows:

$$\begin{aligned}
& \frac{(af, f; q)_\theta}{(aq, f^2/q; q)_\theta} \times \text{LHS} \\
&= \sum_{k \geq 0} \frac{(af, f; q)_\theta}{(aq, f^2/q; q)_\theta} \frac{(a, q^{-\theta}, q^\theta af; q)_k}{(q, q^{-\theta}q/f, q^\theta aq; q)_k} \frac{(a_1, \dots, a_r; q)_k}{(\frac{aq}{a_1}, \dots, \frac{aq}{a_r}; q)_k} \frac{(aq/f; q)_{2k}}{(af; q)_{2k}} \frac{1 - aq^{2k}}{1 - a} z^k \\
&= \sum_{k \geq 0} \frac{(f, q^{2k}af; q)_{\theta-k}}{(f^2/q, q^{2k}aq; q)_{\theta-k}} \frac{(a, q^{-\theta}; q)_k}{(q, q^{-\theta}q^2/f^2; q)_k} \frac{(a_1, \dots, a_r; q)_k}{(\frac{aq}{a_1}, \dots, \frac{aq}{a_r}; q)_k} \frac{(aq/f; q)_{2k}}{(aq; q)_{2k}} \frac{1 - aq^{2k}}{1 - a} (qz/f)^k \\
&= \sum_{k \geq 0} \sum_{\ell \geq 0} \frac{(q^{2k}aq/f, q/f, q^{-\theta+k}; q)_\ell}{(q^{2k}aq, q, q^{-\theta+k}q^2/f^2; q)_\ell} q^\ell \frac{(a, q^{-\theta}; q)_k}{(q, q^{-\theta}q^2/f^2; q)_k} \frac{(a_1, \dots, a_r; q)_k}{(\frac{aq}{a_1}, \dots, \frac{aq}{a_r}; q)_k} \\
&\quad \times \frac{(aq/f; q)_{2k}}{(aq; q)_{2k}} \frac{1 - aq^{2k}}{1 - a} (qz/f)^k \tag{7.7} \\
&= \sum_{m \geq 0} \sum_{k=0}^m \frac{(aq/f, q/f, q^{-\theta}; q)_m}{(aq, q, q^{-\theta}q^2/f^2; q)_m} q^m \frac{(a, q^{-m}, q^m aq/f; q)_k}{(q, q^{-m}f, q^m aq; q)_k} \frac{(a_1, \dots, a_r; q)_k}{(\frac{aq}{a_1}, \dots, \frac{aq}{a_r}; q)_k} \frac{1 - aq^{2k}}{1 - a} z^k \\
&= \sum_{m \geq 0} \frac{(aq/f, q/f, q^{-\theta}; q)_m}{(aq, q, q^{-\theta}q^2/f^2; q)_m} q^m {}_{r+5}W_{r+4}(a; q^{-m}, q^m aq/f, a_1, \dots, a_r; q, z) \\
&= \frac{(af, f; q)_\theta}{(aq, f^2/q; q)_\theta} \times \text{RHS}.
\end{aligned}$$

□

**Lemma 7.4.** *Let  $\theta$  be a nonnegative integer. We have*

$$\begin{aligned}
& \sum_{m \geq 0} \frac{(q/f, q^{-\theta}, aq/f; q)_m}{(q, q^{-\theta}q^2/f^2, aq; q)_m} q^m {}_{10}W_9(a; b, c, d, e, f, g, q^{-m}; q, a^3q^{m+3}/bcdefg) \\
&= \sum_{j \geq 0} \sum_{m \geq j} \frac{(q/f, q^{-\theta}, aq/f; q)_m}{(q, q^{-\theta}q^2/f^2, aq/g; q)_m} q^m \\
&\quad \times \frac{(q^{-m}, f, g, aq/de; q)_j}{(q, aq/d, aq/e, q^{-m}fg/a; q)_j} q^j {}_4\phi_3 \left[ \begin{matrix} q^{-j}, d, e, aq/bc \\ aq/b, aq/c, q^{-j}de/a \end{matrix}; q, q \right]. \tag{7.8}
\end{aligned}$$

Note that this holds also when  $b$  is depending on  $m$ . (In the proof of Theorem 7.1, we will consider that case  $b = q^m aq/f$ .)

*Proof.* Recall Exercise 2.20 (p.53) in Gasper-Rahman [3]:

$$\begin{aligned}
& {}_{10}W_9(a; b, c, d, e, f, g, q^{-n}; q, a^3q^{n+3}/bcdefg) \tag{7.9} \\
&= \frac{(aq, aq/f; q)_n}{(aq/f, aq/g; q)_n} \sum_{j=0}^n \frac{(q^{-n}, f, g, aq/de; q)_j}{(q, aq/d, aq/e, f g q^{-n}/a; q)_j} q^j {}_4\phi_3 \left[ \begin{matrix} q^{-j}, d, e, aq/bc \\ aq/b, aq/c, deq^{-j}/a \end{matrix}; q, q \right].
\end{aligned}$$



Then we have

$$\begin{aligned}
& \text{LHS} \\
&= \sum_{m \geq 0} \sum_{j \geq 0} \frac{(q/f, q^{-\theta_{1,2}}, aq/f; q)_m}{(q, q^{-\theta_{1,2}} q^2/f^2, aq; q)_m} q^m \\
&\times \frac{(aq, aq/f; q)_m}{(aq/f, aq/g; q)_m} \frac{(q^{-m}, f, g, aq/de; q)_j}{(q, aq/d, aq/e, q^{-m}fg/a; q)_j} q^j {}_4\phi_3 \left[ \begin{matrix} q^{-j}, d, e, aq/bc \\ aq/b, aq/c, q^{-j}de/a \end{matrix}; q, q \right] \\
&= \text{RHS}.
\end{aligned} \tag{7.10}$$

Here one should note that we have  $(q^{-m}; q)_j = 0$  when  $m < j$ , and one can exchange the order of the summations as  $\sum_{m \geq 0} \sum_{j \geq 0} = \sum_{j \geq 0} \sum_{m \geq 0} = \sum_{j \geq 0} \sum_{m \geq j}$ .  $\square$

**Proposition 7.5.** *Let  $\theta$  be a nonnegative integer. We have*

$$\begin{aligned}
& \sum_{m \geq 0} \frac{(q/f, q^{-\theta}, aq/f; q)_m}{(q, q^{-\theta} q^2/f^2, aq; q)_m} q^m {}_{10}W_9(a; q^m aq/f, c, d, e, f, af/e, q^{-m}; q, aq^2/cdf) \\
&= \frac{(e, f; q)_\theta}{(eq/f, f^2/q; q)_\theta} \sum_{j \geq 0} (aq^3/cdf^2)^j \frac{(c, d, f, af/e, q^{-\theta}, q^{-\theta}f/e; q)_j}{(q, aq/c, aq/d, aq/e, q^{-\theta+1}/e, q^{-\theta+1}/f; q)_j} \\
&\times {}_5\phi_4 \left[ \begin{matrix} q^{-j}, aq/cd, q/f, q^{\theta-j}f, q^{-j}e/a \\ f, q^{-j+1}/d, q^{-j+1}/c, q^{\theta-j}eq/f \end{matrix}; q, q \right].
\end{aligned} \tag{7.11}$$

*Proof.* In this proof, we use the notations  $b = q^m aq/f, g = af/e$  for simplicity of display. Use the Sears transformation [3] (2.10.4) twice, we have

$$\begin{aligned}
& {}_4\phi_3 \left[ \begin{matrix} q^{-j}, d, e, \frac{aq}{bc} \\ \frac{aq}{b}, \frac{aq}{c}, q^{-j}de/a \end{matrix}; q, q \right] \\
&= \frac{(aq/be, q^{-j}d/a; q)_j}{(aq/b, q^{-j}de/a; q)_j} e^j {}_4\phi_3 \left[ \begin{matrix} q^{-j}, e, \frac{aq}{cd}, b \\ \frac{aq}{c}, q^{-j}\frac{be}{a}, \frac{aq}{d} \end{matrix}; q, q \right] \\
&= \frac{(aq/be, q^{-j}d/a; q)_j}{(aq/b, q^{-j}de/a; q)_j} e^j \frac{(d, c; q)_j}{(aq/c, aq/d; q)_j} (aq/cd)^j {}_4\phi_3 \left[ \begin{matrix} q^{-j}, \frac{aq}{cd}, q^{-j}\frac{b}{a}, q^{-j}\frac{e}{a} \\ q^{-j}\frac{be}{a}, q^{-j}\frac{d}{a}, q^{-j}\frac{c}{a} \end{matrix}; q, q \right].
\end{aligned} \tag{7.12}$$

Use the  $q$ -Saalschütz summation formula, then we have

$$\begin{aligned}
& \sum_{m \geq j} \frac{(q/f, q^{-\theta}, aq/f; q)_m}{(q, q^{-\theta} q^2/f^2, aq/g; q)_m} q^m \frac{(q^{-m}, aq/be; q)_j}{(q^{-m}fg/a, aq/b; q)_j} \frac{(q^{-j}b/a; q)_k}{(q^{-j}be/a; q)_k} \\
&= (q/f^2)^j \frac{(q^{-\theta}; q)_j}{(q^{-\theta} q^2/f^2; q)_j} \frac{(q/f; q)_k}{(aq/g; q)_k} \sum_{m \geq j} \frac{(q^{-\theta+j}, q^k q/f, aq/f; q)_{m-j}}{(q^{-\theta+j} q^2/f^2, q^k aq/g, q; q)_{m-j}} q^{m-j} \\
&= (q/f^2)^j \frac{(q^{-\theta}; q)_j}{(q^{-\theta} q^2/f^2; q)_j} \frac{(q/f; q)_k}{(aq/g; q)_k} \frac{(af/g, q^k f; q)_{\theta-j}}{(f^2/q, q^k aq/g; q)_{\theta-j}}.
\end{aligned} \tag{7.13}$$

Hence from these and Lemma 7.4, we have

LHS

$$\begin{aligned}
&= \sum_{j \geq 0} \sum_{k \geq 0} \sum_{m \geq j} \frac{(q/f, q^{-\theta}, aq/fg; q)_m}{(q, q^{-\theta}q^2/f^2, aq/g; q)_m} q^m \frac{(q^{-m}, f, g, aq/de; q)_j}{(q, aq/d, aq/e, q^{-m}fg/a; q)_j} q^j \\
&\times \frac{(aq/be, dq^{-j}/a; q)_j}{(aq/b, deq^{-j}/a; q)_j} e^j \frac{(d, c; q)_j}{(aq/c, aq/d; q)_j} (aq/cd)^j \\
&\times \frac{(q^{-j}, aq/cd, q^{-j}b/a, q^{-j}e/a; q)_k}{(q, q^{-j}be/a, q^{-j}q/d, q^{-j}q/c; q)_k} q^k \tag{7.14} \\
&= \sum_{j \geq 0} \sum_{k \geq 0} (q/f^2)^j q^j e^j (aq/cd)^j \frac{(q^{-\theta}; q)_j}{(q^{-\theta}q^2/f^2; q)_j} \frac{(q/f; q)_k}{(aq/g; q)_k} \frac{(af/g, q^k f; q)_{\theta-j}}{(f^2/q, q^k aq/g; q)_{\theta-j}} \\
&\times \frac{(f, g, aq/de; q)_j}{(q, aq/d, aq/e; q)_j} \frac{(dq^{-j}/a; q)_j}{(deq^{-j}/a; q)_j} \frac{(d, c; q)_j}{(aq/c, aq/d; q)_j} \\
&\times \frac{(q^{-j}, aq/cd, q^{-j}e/a; q)_k}{(q, q^{-j}q/d, q^{-j}q/c; q)_k} q^k \\
&= \frac{(f, af/g; q)_{\theta}}{(aq/g, f^2/q; q)_{\theta}} \sum_{j \geq 0} (aq^3/cdf^2)^j \frac{(c, d, f, g, q^{-\theta}, q^{-\theta}g/a; q)_j}{(q, aq/c, aq/d, aq/e, q^{-\theta}gq/af, q^{-\theta}q/f; q)_j} \\
&\times {}_5\phi_4 \left[ \begin{matrix} q^{-j}, aq/cd, q/f, q^{\theta-j}f, q^{-j}e/a \\ f, q^{-j+1}/d, q^{-j+1}/c, q^{\theta-j}aq/g \end{matrix}; q, q \right] \\
&= \text{RHS}.
\end{aligned}$$

□

**7.3. Proof of Theorem 7.1.** We specialize the variables  $a, b, c, d, e, f, g$  as

$$\begin{aligned}
a &= q^{-\rho} s_2 / s_1, & aq/b &= q^{1-m}/t, \\
b &= q^{-\rho+m} t s_2 / s_1, & aq/c &= q^{-\rho} t s_2 / s_3, \\
c &= q s_3 / t s_1, & aq/d &= q s_3 / s_1, \\
d &= q^{-\rho} s_2 / s_3, & aq/e &= q^{-\rho} t, \\
e &= q s_2 / t s_1, & aq/f &= q^{-\rho} t s_2 / s_1, \\
f &= q/t, & aq/g &= q s_2 / s_1, \\
g &= q^{-\rho}, & &
\end{aligned} \tag{7.15}$$

in the transformation formulas we have constructed above.

**Lemma 7.6.** *Let  $\theta, \rho$  be nonnegative integers. We have (with (7.15))*

$$\begin{aligned}
&\sum_{k \geq 0} c_3(\theta - k, k, \rho - k; s_1, s_2, s_3 | q, q/t) \\
&= c_3(\theta, 0, \rho; s_1, s_2, s_3 | q, q/t) \\
&\times {}_{14}W_{13}(a; q^{-\theta}, q^{\theta}af, c, d, e, f, g, \left(\frac{aq}{f}\right)^{\frac{1}{2}}, -\left(\frac{aq}{f}\right)^{\frac{1}{2}}, \left(\frac{aq^2}{f}\right)^{\frac{1}{2}}, -\left(\frac{aq^2}{f}\right)^{\frac{1}{2}}; q, t^2).
\end{aligned} \tag{7.16}$$

*Proof.* Straightforward calculation. □

Proposition 7.3, 7.5 and Lemma 7.6 give us the following formula.

**Proposition 7.7.** *Let  $\theta, \rho$  be nonnegative integers. We have*

$$\begin{aligned}
& \sum_{k \geq 0} c_3(\theta - k, k, \rho - k; s_1, s_2, s_3 | q, q/t) \\
&= t^\theta \frac{(q/t, qs_2/ts_1; q)_\theta}{(q, qs_2/s_1; q)_\theta} t^\rho \frac{(q/t, qs_3/ts_2; q)_\rho}{(q, qs_3/s_2; q)_\rho} \\
&\times \sum_{j \geq 0} \frac{(q^{-\theta}, q^{-\theta}s_1/s_2; q)_j}{(q^{-\theta}t, q^{-\theta}ts_1/s_2; q)_j} \frac{(q^{-\rho}, q^{-\rho}s_2/s_3; q)_j}{(q^{-\rho}t, q^{-\rho}ts_2/s_3; q)_j} \frac{(q/t, qs_3/ts_1; q)_j}{(q, qs_3/s_1; q)_j} t^{3j} \\
&\times {}_5\phi_4 \left[ \begin{matrix} t, t, q^{-j}, q^{\theta-j+1}/t, q^{\rho-j+1}/t \\ q/t, q^{-j}ts_1/s_3, q^{\theta-j+1}s_2/s_1, q^{\rho-j+1}s_3/s_2 \end{matrix}; q, q \right].
\end{aligned} \tag{7.17}$$

Now we are ready to state our proof of Theorem 7.1.

*Proof of Theorem 7.1.* The coefficient of the monomial  $(x_2/x_1)^\theta (x_3/x_2)^\rho$  in the series  $p_3(x; s | q, q/t)$  is  $\sum_{k \geq 0} c_3(\theta - k, k, \rho - k; s_1, s_2, s_3 | q, q/t)$ . On the other hand, it can be easily shown that the coefficient of the monomial  $(x_2/x_1)^\theta (x_3/x_2)^\rho$  in the series on RHS of (7.2) is given by the RHS of (7.17). Hence we have the equality (7.2).  $\square$

## 8. FAMILY OF COMMUTATIVE INTEGRAL TRANSFORMATIONS

We revisit the problem considered in [11].

Suppose  $g(y)$  is a joint eigenfunction of Ruijsenaars-Macdonald operators in  $n$  variables  $y = (y_1, \dots, y_n)$

$$D^y(u)g(y) = g(y) \prod_{i=1}^n (1 - ut^{n-i}q^{\lambda_i}) \tag{8.1}$$

with complex parameters  $\lambda = (\lambda_1, \dots, \lambda_n)$ . Define a function  $f(x)$  in the same number of variables  $x = (x_1, \dots, x_n)$  by an integral transformation of the form

$$f(x) = \int \prod_{i,j=1}^n \frac{(tx_j/y_i; q)_\infty}{(x_j/y_i; q)_\infty} \prod_{1 \leq i, j \leq n; i \neq j} \frac{(y_i/y_j; q)_\infty}{(ty_i/y_j; q)_\infty} c(x, y) g(y) d\omega(y), \tag{8.2}$$

where  $d\omega(y)$  is a  $q$ -invariant measure on  $\mathbb{T}_y^n$ ,  $c(x, y) = c(x_1, \dots, x_n, y_1, \dots, y_n)$  satisfies the conditions

$$T_{q, x_i} c(x, y) = \alpha c(x, y), \quad T_{q, y_i} c(x, y) = \alpha^{-1} c(x, y) \quad (1 \leq i \leq n), \tag{8.3}$$

where  $\alpha \in \mathbb{C}^*$  is a parameter. One of the choice of such  $c(x, y)$  is

$$c(x, y) = (x/y)^\lambda \prod_{i=1}^n \frac{\theta(\alpha q y_i / s_i t x_i; q)}{\theta(q y_i / t x_i; q)} \prod_{1 \leq i < j \leq n} \frac{\theta(q y_j / x_i; q) \theta(q y_j / t y_i; q)}{\theta(q y_j / t x_i; q) \theta(q y_j / y_i; q)}, \tag{8.4}$$

where  $s_i = t^{n-i}q^{\lambda_i}$  ( $i = 1, \dots, n$ ). Then by using the same argument as in [7], we can conclude that we have

$$D^x(u)f(x) = f(x) \prod_{j=1}^n (1 - ut^{n-j}q^{\lambda_j}). \tag{8.5}$$

Let  $t$  satisfies  $|t| > 1$ , and fix  $r$  such that  $0 < r < 1$ . Consider a function  $f(x) = x^\lambda \varphi(x)$  with  $\varphi(x)$  being a holomorphic function in the variables  $(x_2/x_1, \dots, x_n/x_{n-1})$  defined on  $|x_{i+1}/x_i| < r$  ( $i = 1, \dots, n-1$ ). Let  $\alpha \in \mathbb{C}^*$  be a parameter, and set

$$\begin{aligned} \psi(x) = & \int_{C_1} \cdots \int_{C_n} \prod_{k=1}^n \frac{dy_k}{2\pi i y_k} \prod_{\ell=1}^n \frac{\theta(\alpha q y_\ell / s_\ell t x_\ell; q)}{\theta(q y_\ell / t x_\ell; q)} \frac{(t x_\ell / y_\ell; q)_\infty}{(x_\ell / y_\ell; q)_\infty} \\ & \times \prod_{1 \leq i < j \leq n} \frac{(t x_j / y_i; q)_\infty}{(x_j / y_i; q)_\infty} \frac{(q y_j / x_i; q)_\infty}{(q y_j / t x_i; q)_\infty} (1 - y_j / y_i) \frac{(q y_j / t y_i; q)_\infty}{(t y_j / y_i; q)_\infty} \varphi(y), \end{aligned} \quad (8.6)$$

where the contours  $C_i$  are defined by  $|y_i| = a_i$  with  $|q^{-1} t x_i| > a_i > |x_i|$  ( $i = 1, \dots, n$ ). Then  $\psi(x)$  is a holomorphic function in the variables  $(x_2/x_1, \dots, x_n/x_{n-1})$  on  $|x_{i+1}/x_i| < |q/t| r$  ( $i = 1, \dots, n-1$ ). Since  $|t| > 1$ , our integration cycle is  $q$ -shift invariant. We define the integral transform  $(I(\alpha)f)(x)$  of  $f(x) = x^\lambda \varphi(x)$  by setting  $(I(\alpha)f)(x) = x^\lambda \psi(x)$ .

**Theorem 8.1.** *For any  $\alpha, \beta \in \mathbb{C}^*$ , we have*

$$[D^x(u), I(\alpha)] = 0, \quad (8.7)$$

$$[I(\alpha), I(\beta)] = 0. \quad (8.8)$$

*Proof.* We have (8.7) since  $I(\alpha)(x^\lambda p_n(x; s|q, t))$  is proportional to  $x^\lambda p_n(x; s|q, t)$ . Then (8.8) follows from (8.7).  $\square$

## REFERENCES

- [1] I. Cherednik: Difference Macdonald-Mehta conjecture, Int. Math. Res. Not. IMRN 1997, 449–467.
- [2] I. Cherednik: Whittaker limits of difference spherical functions, Int. Math. Res. Not. IMRN 2009, 3793–3842.
- [3] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, (1990).
- [4] I.G. Macdonald: *Symmetric Functions and Hall Polynomials*, Second Edition, Oxford Mathematical Monographs, Oxford University Press, 1995.
- [5] I.G. Macdonald: Affine Hecke algebras and orthogonal polynomials, Cambridge Tracts in Mathematics **153**, Cambridge University Press, 2003.
- [6] M. van Meer and J. Stokman: Double affine Hecke algebras and bispectral quantum Knizhnik-Zamolodchikov equations, Int. Math. Res. Not. IMRN 2010, 969–1040.
- [7] K. Mimachi and M. Noumi: An integral representation of eigenfunctions for Macdonald’s  $q$ -difference operators, Tôhoku Math. J. **49** (1997), 517–525.
- [8] M. Noumi and A. Sano: An infinite family of higher-order difference operators that commute with Ruijsenaars operators of type  $A$ , in preparation.
- [9] R.N.M. Ruijsenaars: Complete integrability of relativistic Calogero-Moser systems and elliptic function identities, Commun. Math. Phys. **110** (1987), 191–213.
- [10] J. Shiraishi: A conjecture about raising operators for Macdonald polynomials, Lett. Math. Phys. **73** (2005), 71–81.
- [11] J. Shiraishi: A family of integral transformations and basic hypergeometric series, Commun. Math. Phys. **263** (2006), 439–460.
- [12] J. Stokman: The  $c$ -function expansion of a basic hypergeometric function associated to root systems, arXiv:1109.0613.

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